IberoAmerican Olympiad For University Students 2006

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1 Let m,n be positive integers greater than 1. We define the sets $P_m=\{\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}\}$ and $P_n=\{\frac{1}{n},\frac{2}{n},\cdots,\frac{n-1}{n}\}$.

Find the distance between P_m and P_n, that is defined as

$$\min\{|a-b|:a\in P_m,b\in P_n\}$$

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2 Prove that for any positive integer n and any real numbers $a_1,a_2,\cdots,a_n,b_1,b_2,\cdots,b_n$ we have that the equation

$$a_1\sin(x)+a_2\sin(2x)+\cdots+a_n\sin(nx)=b_1\cos(x)+b_2\cos(2x)+\cdots+b_n\cos(nx)$$

has at least one real root.

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3 Let $p_1(x)=p(x)=4x^3-3x$ and $p_{n+1}(x)=p(p_n(x))$ for each positive integer n. Also, let $A(n)$ be the set of all the real roots of the equation $p_n(x)=x$.

Prove that $A(n)\subseteq A(2n)$ and that the product of the elements of $A(n)$ is the average of the elements of $A(2n)$.

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4 Prove that for any interval $[a,b]$ of real numbers and any positive integer n there exists a positive integer k and a partition of the given interval

$$a=x(0)<x(1)<x(2)<\cdots<x(k-1)<x(k)=b$$

such that

$$\int_{x(0)}^{x(1)}f(x)dx+\int_{x(2)}^{x(3)}f(x)dx+\cdots=\int_{x(1)}^{x(2)}f(x)dx+\int_{x(3)}^{x(4)}f(x)dx+\cdots$$

for all polynomials f with real coefficients and degree less than n.

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5 A regular n-gon is inscribed in a circle of radius 1. Let $a_1,\cdots,a_{n-1}$ be the distances of one of the vertices of the polygon to all the other vertices. Prove that

$$(5-a_1^2)\cdots(5-a_{n-1}^2)=F_n^2$$

where F_n is the n^{th} term of the Fibonacci sequence $1,1,2,\cdots$

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6 Let $x_0(t)=1$, $x_{k+1}(t)=(1+t^{k+1})x_k(t)$ for all $k\geq 0$; $y_{n,0}(t)=1$, $y_{n,k}(t)=\frac{t^{n-k+1}-1}{t^k-1}y_{n,k-1}(t)$ for all $n\geq 0$, $1\leq k\leq n$.

Prove that $\sum_{j=0}^{n-1}(-1)^j x_{n-j-1}(t)y_{n,j}(t)=\frac{1-(-1)^n}{2}$ for all $n\geq 1$.

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7 Consider the multiplicative group $A=\{z\in\mathbb{C}|z^{2006^k}=1, 0<k\in\mathbb{Z}\}$ of all the roots of unity of degree 2006^k for all positive integers k.

Find the number of homomorphisms $f:A\to A$ that satisfy $f(f(x))=f(x)$ for all elements $x\in A$.
