IberoAmerican Olympiad For University Students 2007

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1 For each pair of integers (i,k) such that $1 \le i \le k$, the linear transformation $P_{i,k}:\mathbb{R}^k \to \mathbb{R}^k$ is defined as:

$P_{i,k}(a_1,\cdots,a_{i-1},a_i,a_{i+1},\cdots,a_k)=(a_1,\cdots,a_{i-1},0,a_{i+1},\cdots,a_k)$

Prove that for all $n \ge 2$ and for every set of $n-1$ linearly independent vectors $v_1,\cdots,v_{n-1}$ in $\mathbb{R}^n$, there is an integer k such that $1 \le k \le n$ and such that the vectors $P_{k,n}(v_1),\cdots,P_{k,n}(v_{n-1})$ are linearly independent.

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2 Prove that for all positive integers n and for all real numbers x such that $0 \le x \le 1$, the following inequality holds: $\left(1-x+\frac{x^2}{2}\right)^n-(1-x)^n \le \frac{x}{2}$.

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3 Let $f:\mathbb{R}\to\mathbb{R}^+$ be a continuous and periodic function. Prove that for all $\alpha \in \mathbb{R}$ the following inequality holds:

$\int_0^T \frac{f(x)}{f(x+\alpha)}dx \ge T$,

where T is the period of $f(x)$.

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4 Consider an infinite sequence $a_1,a_2,\cdots$ whose terms all belong to $\{1,2\}$. A positive integer with n digits is said to be *good* if its decimal representation has the form $a_ra_{r+1}\cdots a_{r+(n-1)}$, for some positive integer r. Suppose that there are at least 2008 *good* numbers with a million digits. Prove that there are at least 2008 *good* numbers with 2007 digits.

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5 Determine all pairs of polynomials $f,g\in\mathbb{C}[x]$ with complex coefficients such that the following equalities hold for all $x\in\mathbb{C}$:

$f(f(x))-g(g(x))=1+i\ f(g(x))-g(f(x))=1-i$

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6 Let F be a field whose characteristic is not 2, let $F^*=F\setminus\{0\}$ be its multiplicative group and let T be the subgroup of F^* constituted by its finite order elements. Prove that if T is finite, then T is cyclic and its order is even.

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7 The *height* of a positive integer is defined as being the fraction $\frac{s(a)}{a}$, where $s(a)$ is the sum of all the positive divisors of a. Show that for every pair of positive integers N,k there is a positive integer b such that the *height* of each of $b,b+1,\cdots,b+k$ is greater than N.
