IberoAmerican Olympiad For University Students 2008

www.artofproblemsolving.com/community/c3983

by Jorge Miranda

1 Let n be a positive integer that is not divisible by either 2 or 5.
In the decimal expansion of $\frac{1}{n}=0.a_1a_2a_3\cdots$ a finite number of digits after the decimal point are chosen arbitrarily to be deleted.
Clearly the decimal number obtained by this procedure is also rational, so it's equal to $\frac{a}{b}$ for some integers a,b. Prove that b is divisible by n.

---

2 Prove that for each natural number n there is a polynomial f with real coefficients and degree n such that $p(x)=f(x^2-1)$ is divisible by $f(x)$ over the ring $\mathbb{R}[x]$.

---

3 Prove that $x+\frac{1}{x^x}<2$ for $0<x<1$.

---

4 Two vertices A,B of a triangle ABC are located on a parabola $y=ax^2+bx+c$ with $a>0$ in such a way that the sides AC,BC are tangent to the parabola.
Let m_c be the length of the median CC_1 of triangle ABC and S be the area of triangle ABC. Find

$$\frac{S^2}{m_c^3}$$

---

5 Find all positive integers n such that there are positive integers $a_1,\cdots,a_n,b_1,\cdots,b_n$ that satisfy

$$(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2)-(a_1b_1+\cdots+a_nb_n)^2=n$$

---

6 ***a)*** Determine if there are matrices $A,B,C\in\mathrm{SL}_2(\mathbb{Z})$ such that $A^2+B^2=C^2$.

b) Determine if there are matrices $A,B,C\in\mathrm{SL}_2(\mathbb{Z})$ such that $A^4+B^4=C^4$.

Note: The notation $A\in\mathrm{SL}_2(\mathbb{Z})$ means that A is a 2×2 matrix with integer entries and $\det A=1$.

---

7 Let A be an abelian additive group such that all nonzero elements have infinite order and for each prime number p we have the inequality $|A/pA|\le p$, where $pA=\{pa|a\in A\}$, $pa=a+a+\cdots+a$ (where the sum has p summands) and $|A/pA|$ is the order of the quotient group A/pA (the index of the subgroup pA).

Prove that each subgroup of A of finite index is isomorphic to A.
