IberoAmerican Olympiad For University Students 2009

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by Jorge Miranda

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1 A line through a vertex of a non-degenerate triangle cuts it in two similar triangles with $\sqrt{3}$ as the ratio between correspondent sides.
Find the angles of the given triangle.

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2 Let $x_1, \cdots, x_n$ be nonzero vectors of a vector space V and $\varphi : V \to V$ be a linear transformation such that $\varphi x_1 = x_1$, $\varphi x_k = x_k - x_{k-1}$ for $k = 2, 3, \ldots, n$.
Prove that the vectors $x_1, \ldots, x_n$ are linearly independent.

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3 Let $a, b, c, d, e \in \mathbb{R}^+$ and $f : \{(x, y) \in (\mathbb{R}^+)^2 | c - dx - ey > 0\} \to \mathbb{R}^+$ be given by $f(x, y) = (ax)(by)(c - dx - ey)$.
Find the maximum value of f.

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4 Given two positive integers m, n, we say that a function $f : [0, m] \to \mathbb{R}$ is (m, n)-*slippery* if it has the following properties:

i) f is continuous;
ii) $f(0) = 0$, $f(m) = n$;
iii) If $t_1, t_2 \in [0, m]$ with $t_1 < t_2$ are such that $t_2 - t_1 \in \mathbb{Z}$ and $f(t_2) - f(t_1) \in \mathbb{Z}$, then $t_2 - t_1 \in \{0, m\}$.

Find all the possible values for m, n such that there is a function f that is (m, n)-slippery.

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5 Let $\mathbb{N}$ and $\mathbb{N}^*$ be the sets containing the natural numbers/positive integers respectively.

We define a binary relation on $\mathbb{N}$ by $a \acute{\in} b$ iff the a-th bit in the binary representation of b is 1.

We define a binary relation on $\mathbb{N}^*$ by $a \tilde{\in} b$ iff b is a multiple of the a-th prime number p_a.

i) Prove that there is no bijection $f : \mathbb{N} \to \mathbb{N}^*$ such that $a \acute{\in} b \Leftrightarrow f(a) \tilde{\in} f(b)$.
ii) Prove that there is a bijection $g : \mathbb{N} \to \mathbb{N}^*$ such that $(a \acute{\in} b \vee b \acute{\in} a) \Leftrightarrow (g(a) \tilde{\in} g(b) \vee g(b) \tilde{\in} g(a))$.

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6 Let $\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_e \in \mathbb{C}$ be such that the polynomials

$f_1(x) = \prod_{i=1}^d (x - \alpha_i)$ and $f_2(x) = \prod_{i=1}^e (x - \beta_i)$

have integer coefficients.

Suppose that there exist polynomials $g_1, g_2 \in \mathbb{Z}[x]$ such that $f_1 g_1 + f_2 g_2 = 1$.

Prove that $\left| \prod_{i=1}^d \prod_{j=1}^e (\alpha_i - \beta_j) \right| = 1$

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7 Let G be a group such that every subgroup of G is subnormal. Suppose that there exists N normal subgroup of G such that $Z(N)$ is nontrivial and G/N is cyclic. Prove that $Z(G)$ is non-trivial. ($Z(G)$ denotes the center of G).

Note: A subgroup H of G is subnormal if there exist subgroups $H_1, H_2, \ldots, H_m = G$ of G such that $H \lhd H_1 \lhd H_2 \lhd \ldots \lhd H_m = G$ ($\lhd$ denotes normal subgroup).