IberoAmerican Olympiad For University Students 2010

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by Joao Pedro Santos

1 Let $f : S \to \mathbb{R}$ be the function from the set of all right triangles into the set of real numbers, defined by $f(\Delta ABC) = \frac{h}{r}$, where h is the height with respect to the hypotenuse and r is the inscribed circle's radius. Find the image, $Im(f)$, of the function.

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2 Calculate the sum of the series $\sum_{-\infty}^{\infty} \frac{\sin^3 3^k}{3^k}$.

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3 A student adds up rational fractions incorrectly:

$$\frac{a}{b} + \frac{x}{y} = \frac{a+x}{b+y} \qquad (\star)$$

Despite that, he sometimes obtains correct results. For a given fraction $\frac{a}{b}, a, b \in \mathbb{Z}, b > 0$, find all fractions $\frac{x}{y}, x, y \in \mathbb{Z}, y > 0$ such that the result obtained by $(\star)$ is correct.

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4 Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a monic polynomial of degree $n > 2$, with real coefficients and all its roots real and different from zero. Prove that for all $k = 0, 1, 2, \cdots, n-2$, at least one of the coefficients a_k, a_{k+1} is different from zero.

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5 Let A, B be matrices of dimension 2010×2010 which commute and have real entries, such that $A^{2010} = B^{2010} = I$, where I is the identity matrix. Prove that if $\operatorname{tr}(AB) = 2010$, then $\operatorname{tr}(A) = \operatorname{tr}(B)$.

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6 Prove that, for all integer $a > 1$, the prime divisors of $5a^4 - 5a^2 + 1$ have the form $20k \pm 1, k \in \mathbb{Z}$.

Proposed by Gza Ks.

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7 (a) Prove that, for any positive integers $m \leq \ell$ given, there is a positive integer n and positive integers $x_1, \cdots, x_n, y_1, \cdots, y_n$ such that the equality

$$\sum_{i=1}^{n} x_i^k = \sum_{i=1}^{n} y_i^k$$

holds for every $k = 1, 2, \cdots, m-1, m+1, \cdots, \ell$, but does not hold for $k = m$.

(b) Prove that there is a solution of the problem, where all numbers $x_1, \cdots, x_n, y_1, \cdots, y_n$ are distinct.

Proposed by Ilya Bogdanov and Gza Ks.

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