**Benelux 2009**

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by WakeUp

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| – | May 9th |
| 1 | Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ that satisfy the following two conditions: • $f(n)$ is a perfect square for all $n \in \mathbb{Z}_{>0}$ • $f(m+n) = f(m) + f(n) + 2mn$ for all $m, n \in \mathbb{Z}_{>0}$. |
| 2 | Let $n$ be a positive integer and let $k$ be an odd positive integer. Moreover, let $a, b$ and $c$ be integers (not necessarily positive) satisfying the equations $$a^n + kb = b^n + kc = c^n + ka$$ Prove that $a = b = c$. |
| 3 | Let $n \geq 1$ be an integer. In town $X$ there are $n$ girls and $n$ boys, and each girl knows each boy. In town $Y$ there are $n$ girls, $g_1, g_2, \ldots, g_n$, and $2n-1$ boys, $b_1, b_2, \ldots, b_{2n-1}$. For $i = 1, 2, \ldots, n$, girl $g_i$ knows boys $b_1, b_2, \ldots, b_{2i-1}$ and no other boys. Let $r$ be an integer with $1 \leq r \leq n$. In each of the towns a party will be held where $r$ girls from that town and $r$ boys from the same town are supposed to dance with each other in $r$ dancing pairs. However, every girl only wants to dance with a boy she knows. Denote by $X(r)$ the number of ways in which we can choose $r$ dancing pairs from town $X$, and by $Y(r)$ the number of ways in which we can choose $r$ dancing pairs from town $Y$. Prove that $X(r) = Y(r)$ for $r = 1, 2, \ldots, n$. |
| 4 | Given trapezoid $ABCD$ with parallel sides $AB$ and $CD$, let $E$ be a point on line $BC$ outside segment $BC$, such that segment $AE$ intersects segment $CD$. Assume that there exists a point $F$ inside segment $AD$ such that $\angle EAD = \angle CBF$. Denote by $I$ the point of intersection of $CD$ and $EF$, and by $J$ the point of intersection of $AB$ and $EF$. Let $K$ be the midpoint of segment $EF$, and assume that $K$ is different from $I$ and $J$. <br><br> Prove that $K$ belongs to the circumcircle of $\triangle ABI$ if and only if $K$ belongs to the circumcircle of $\triangle CDJ$. |