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1	Let $n \geq 3$ be an integer. A frog is to jump along the real axis, starting at the point 0 and making n jumps: one of length 1, one of length 2, $\dots$, one of length n. It may perform these n jumps in any order. If at some point the frog is sitting on a number $a \leq 0$, its next jump must be to the right (towards the positive numbers). If at some point the frog is sitting on a number $a > 0$, its next jump must be to the left (towards the negative numbers). Find the largest positive integer k for which the frog can perform its jumps in such an order that it never lands on any of the numbers $1, 2, \dots, k$.
2	Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $$f(x+y)+y \leq f(f(f(x)))$$ holds for all $x, y \in \mathbb{R}$.
3	Let $\triangle ABC$ be a triangle with circumcircle Γ, and let I be the center of the incircle of $\triangle ABC$. The lines AI, BI and CI intersect Γ in $D \neq A$, $E \neq B$ and $F \neq C$. The tangent lines to Γ in F, D and E intersect the lines AI, BI and CI in R, S and T, respectively. Prove that $$\vert AR\vert \cdot \vert BS\vert \cdot \vert CT\vert = \vert ID\vert \cdot \vert IE\vert \cdot \vert IF\vert.$$
4	a) Find all positive integers g with the following property: for each odd prime number p there exists a positive integer n such that p divides the two integers $$g^n - n \quad \text{and} \quad g^{n+1} - (n+1).$$ b) Find all positive integers g with the following property: for each odd prime number p there exists a positive integer n such that p divides the two integers $$g^n - n^2 \quad \text{and } g^{n+1} - (n+1)^2.$$