Spain Mathematical Olympiad 2000

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Day 1

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1 Consider the polynomials

$$P(x) = x^4 + ax^3 + bx^2 + cx + 1 \quad \text{and} \quad Q(x) = x^4 + cx^3 + bx^2 + ax + 1.$$

Find the conditions on the parameters a, b,c with $a \neq c$ for which $P(x)$ and $Q(x)$ have two common roots and, in such cases, solve the equations $P(x) = 0$ and $Q(x) = 0$.

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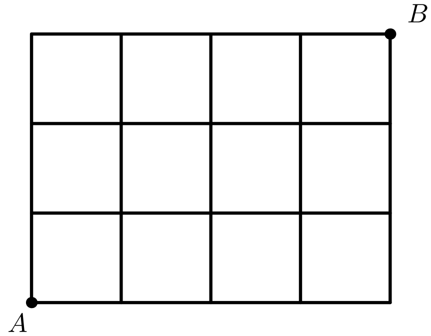
2 The figure shows a network of roads bounding 12 blocks. Person P goes from A to B, and person Q goes from B to A, each going by a shortest path (along roads). The persons start simultaneously and go at the same constant speed. At each point with two possible directions to take, both have the same probability. Find the probability that the persons meet.



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3 Two circles C_1 and C_2 with the respective radii r_1 and r_2 intersect in A and B. A variable line r through B meets C_1 and C_2 again at P_r and Q_r respectively. Prove that there exists a point M, depending only on C_1 and C_2, such that the perpendicular bisector of each segment P_rQ_r passes through M.

Day 2

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1 Find the largest integer N satisfying the following two conditions:

(i) $\left[\frac{N}{3}\right]$ consists of three equal digits;

(ii) $\left[\frac{N}{3}\right] = 1 + 2 + 3 + \cdots + n$ for some positive integer n.

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2 Four points are given inside or on the boundary of a unit square. Prove that at least two of these points are on a mutual distance at most 1.

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3 Show that there is no function $f:\mathbb{N}\to\mathbb{N}$ satisfying $f(f(n))=n+1$ for each positive integer n.
