Spain Mathematical Olympiad 2010

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Day 1

1 A *pucelana* sequence is an increasing sequence of 16 consecutive odd numbers whose sum is a perfect cube. How many pucelana sequences are there with 3-digit numbers only?

2 Let $\mathbb{N}_0$ and $\mathbb{Z}$ be the set of all non-negative integers and the set of all integers, respectively. Let $f:\mathbb{N}_0\rightarrow\mathbb{Z}$ be a function defined as

$$f(n)=-f\left(\left\lfloor\frac{n}{3}\right\rfloor\right)-3\left\{\frac{n}{3}\right\}$$

where $\lfloor x\rfloor$ is the greatest integer smaller than or equal to x and $\{x\}=x-\lfloor x\rfloor$. Find the smallest integer n such that $f(n)=2010$.

3 Let $ABCD$ be a convex quadrilateral. AC and BD meet at P, with $\angle APD=60^{\circ}$. Let E,F,G, and H be the midpoints of AB,BC,CD and DA respectively. Find the greatest positive real number k for which

$$EG+3HF\geq kd+(1-k)s$$

where s is the semi-perimeter of the quadrilateral $ABCD$ and d is the sum of the lengths of its diagonals. When does the equality hold?

Day 2

1 Let a,b,c be three positive real numbers. Show that

$$\frac{a+b+3c}{3a+3b+2c}+\frac{a+3b+c}{3a+2b+3c}+\frac{3a+b+c}{2a+3b+3c}\geq\frac{15}{8}$$

2 In a triangle ABC, let P be a point on the bisector of $\angle BAC$ and let A',B' and C' be points on lines BC,CA and AB respectively such that PA' is perpendicular to $BC,PB'\perp AC$, and $PC'\perp AB$. Prove that PA' and $B'C'$ intersect on the median AM, where M is the midpoint of BC.

3 Let p be a prime number and A an infinite subset of the natural numbers. Let $f_A(n)$ be the number of different solutions of $x_1+x_2+\ldots+x_p=n$, with $x_1,x_2,\ldots x_p\in A$. Does there exist a number N for which $f_A(n)$ is constant for all $n<N$?