Spain Mathematical Olympiad 2011

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Day 1

1 Each pair of vertices of a regular 67-gon is joined by a line segment. Suppose n of these segments are selected, and each of them is painted one of ten available colors. Find the minimum possible value of n for which, regardless of which n segments were selected and how they were painted, there will always be a vertex of the polygon that belongs to seven segments of the same color.

2 Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}\geq\frac{5}{2}$$

and determine when equality holds.

3 Let A, B, C, D be four points in space not all lying on the same plane. The segments AB, BC, CD, and DA are tangent to the same sphere. Prove that their four points of tangency are coplanar.

Day 2

1 In triangle ABC, $\angle B = 2\angle C$ and $\angle A > 90^\circ$. Let D be the point on the line AB such that CD is perpendicular to AC, and let M be the midpoint of BC. Prove that $\angle AMB = \angle DMC$.

2 Each rational number is painted either white or red. Call such a coloring of the rationals *sanferminera* if for any distinct rationals numbers x and y satisfying one of the following three conditions: -$xy = 1$,
-$x + y = 0$,
-$x + y = 1$,we have x and y painted different colors. How many sanferminera colorings are there?

3 The sequence $S_0, S_1, S_2, \ldots$ is defined by-$S_n = 1$ for $0 \leq n \leq 2011$, and
-$S_{n+2012} = S_{n+2011} + S_n$ for $n \geq 0$.Prove that $S_{2011a} - S_a$ is a multiple of 2011 for all nonnegative integers a.