Spain Mathematical Olympiad 2012

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by WakeUp

Day 1

1 Determine if the number $\lambda_n=\sqrt{3n^2+2n+2}$ is irrational for all non-negative integers n.

2 Find all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that

$$(x-2)f(y)+f(y+2f(x))=f(x+yf(x))$$

for all $x,y\in\mathbb{R}$.

3 Let x and n be integers such that $1\le x\le n$. We have $x+1$ separate boxes and $n-x$ identical balls. Define $f(n,x)$ as the number of ways that the $n-x$ balls can be distributed into the $x+1$ boxes. Let p be a prime number. Find the integers n greater than 1 such that the prime number p is a divisor of $f(n,x)$ for all $x\in\{1,2,\ldots ,n-1\}$.

Day 2

1 Find all positive integers n and k such that $(n+1)^n=2n^k+3n+1$.

2 A sequence $(a_n)_{n\ge 1}$ of integers is defined by the recurrence

$$a_1=1,\ a_2=5,\ a_n=\frac{a_{n-1}^2+4}{a_{n-2}}\text{ for }n\ge 2.$$

Prove that all terms of the sequence are integers and find an explicit formula for a_n.

3 Let ABC be an acute-angled triangle. Let ω be the inscribed circle with centre I, Ω be the circumscribed circle with centre O and M be the midpoint of the altitude AH where H lies on BC. The circle ω be tangent to the side BC at the point D. The line MD cuts ω at a second point P and the perpendicular from I to MD cuts BC at N. The lines NR and NS are tangent to the circle Ω at R and S respectively. Prove that the points R,P,D and S lie on the same circle.