Spain Mathematical Olympiad 2014

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Day 1

1 Is it possible to place the numbers $0, 1, 2, \ldots, 9$ on a circle so that the sum of any three consecutive numbers is a) 13, b) 14, c) 15?

2 Given the rational numbers r, q, and n, such that $\frac{1}{r+qn} + \frac{1}{q+rn} = \frac{1}{r+q}$, prove that $\sqrt{\frac{n-3}{n+1}}$ is a rational number.

3 Let B and C be two fixed points on a circle centered at O that are not diametrically opposed. Let A be a variable point on the circle distinct from B and C and not belonging to the perpendicular bisector of BC. Let H be the orthocenter of $\triangle ABC$, and M and N be the midpoints of the segments BC and AH, respectively. The line AM intersects the circle again at D, and finally, NM and OD intersect at P. Determine the locus of points P as A moves around the circle.

Day 2

1 Let (x_n) be a sequence of positive integers defined by $x_1 = 2$ and $x_{n+1} = 2x_n^3 + x_n$ for all integers $n \geq 1$. Determine the largest power of 5 that divides $x_{2014}^2 + 1$.

2 Let M be the set of all integers in the form of $a^2 + 13b^2$, where a and b are distinct itnegers.

i) Prove that the product of any two elements of M is also an element of M.

ii) Determine, reasonably, if there exist infinite pairs of integers (x, y) so that $x + y \notin M$ but $x^{13} + y^{13} \in M$.

3 60 points are on the interior of a unit circle (a circle with radius 1). Show that there exists a point V on the circumference of the circle such that the sum of the distances from V to the 60 points is less than or equal to 80.