IMO Longlists 1966

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1 Given $n > 3$ points in the plane such that no three of the points are collinear. Does there exist a circle passing through (at least) 3 of the given points and not containing any other of the n points in its interior ?

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2 Given n positive numbers $a_1, a_2, ..., a_n$ such that $a_1 \cdot a_2 \cdot ... \cdot a_n = 1$. Prove

$$(1 + a_1)(1 + a_2)...(1 + a_n) \geq 2^n.$$

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3 A regular triangular prism has the altitude $h,$ and the two bases of the prism are equilateral triangles with side length $a.$ Dream-holes are made in the centers of both bases, and the three lateral faces are mirrors. Assume that a ray of light, entering the prism through the dream-hole in the upper base, then being reflected once by any of the three mirrors, quits the prism through the dream-hole in the lower base. Find the angle between the upper base and the light ray at the moment when the light ray entered the prism, and the length of the way of the light ray in the interior of the prism.

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4 Given 5 points in the plane, no three of them being collinear. Show that among these 5 points, we can always find 4 points forming a convex quadrilateral.

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5 Prove the inequality

$$\tan \frac{\pi \sin x}{4 \sin \alpha} + \tan \frac{\pi \cos x}{4 \cos \alpha} > 1$$

for any x, α with $0 \leq x \leq \frac{\pi}{2}$ and $\frac{\pi}{6} < \alpha < \frac{\pi}{3}.$

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6 Let m be a convex polygon in a plane, l its perimeter and S its area. Let $M(R)$ be the locus of all points in the space whose distance to m is $\leq R,$ and $V(R)$ is the volume of the solid $M(R).$

a.) Prove that

$$V(R) = \frac{4}{3}\pi R^3 + \frac{\pi}{2} l R^2 + 2SR.$$

Hereby, we say that the distance of a point C to a figure m is $\leq R$ if there exists a point D of the figure m such that the distance CD is $\leq R.$ (This point D may lie on the boundary of the figure m and inside the figure.)

additional question:

b.) Find the area of the planar R-neighborhood of a convex or non-convex polygon m.

c.) Find the volume of the R-neighborhood of a convex polyhedron, e. g. of a cube or of a tetrahedron.

Note by Darij: I guess that the "R-neighborhood" of a figure is defined as the locus of all points whose distance to the figure is $\leq R$.

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7 For which arrangements of two infinite circular cylinders does their intersection lie in a plane?

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8 We are given a bag of sugar, a two-pan balance, and a weight of 1 gram. How do we obtain 1 kilogram of sugar in the smallest possible number of weighings?

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9 Find x such that trigonometric

$$\frac{\sin 3x \cos(60^\circ - x) + 1}{\sin(60^\circ - 7x) - \cos(30^\circ + x) + m} = 0$$

where m is a fixed real number.

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10 How many real solutions are there to the equation $x = 1964 \sin x - 189$?

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11 Does there exist an integer z that can be written in two different ways as $z = x! + y!$, where x, y are natural numbers with $x \leq y$?

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12 Find digits x, y, z such that the equality

$$\sqrt{\underbrace{\overline{xx\cdots x}}_{2n \text{ times}} - \underbrace{\overline{yy\cdots y}}_{n \text{ times}}} = \underbrace{\overline{zz\cdots z}}_{n \text{ times}}$$

holds for at least two values of $n \in \mathbb{N}$, and in that case find all n for which this equality is true.

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13 Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove the inequality

$$\binom{n}{2} \sum_{i<j} \frac{1}{a_i a_j} \geq 4 \left(\sum_{i<j} \frac{1}{a_i + a_j} \right)^2$$

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14 What is the maximal number of regions a circle can be divided in by segments joining n points on the boundary of the circle ?

Posted already on the board I think...

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15 Given four points A, B, C, D on a circle such that AB is a diameter and CD is not a diameter. Show that the line joining the point of intersection of the tangents to the circle at the points C and D with the point of intersection of the lines AC and BD is perpendicular to the line AB.

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16 We are given a circle K with center S and radius 1 and a square Q with center M and side 2. Let XY be the hypotenuse of an isosceles right triangle XYZ. Describe the locus of points Z as X varies along K and Y varies along the boundary of Q.

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17 Let $ABCD$ and $A'B'C'D'$ be two arbitrary parallelograms in the space, and let M, N, P, Q be points dividing the segments AA', BB', CC', DD' in equal ratios.

a.) Prove that the quadrilateral $MNPQ$ is a parallelogram.

b.) What is the locus of the center of the parallelogram $MNPQ$, when the point M moves on the segment AA' ?

(Consecutive vertices of the parallelograms are labelled in alphabetical order.

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18 Solve the equation $\frac{1}{\sin x} + \frac{1}{\cos x} = \frac{1}{p}$, where p is a real parameter. Investigate for which values of p solutions exist and how many solutions exist.

(Of course, the last question "how many solutions exist" should be understood as "how many solutions exists modulo 2π".)

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19 Construct a triangle given the radii of the excircles.

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20 Given three congruent rectangles in the space. Their centers coincide, but the planes they lie in are mutually perpendicular. For any two of the three rectangles, the line of intersection of the planes of these two rectangles contains one midparallel of one rectangle and one midparallel of the other rectangle, and these two midparallels have different lengths. Consider the convex polyhedron whose vertices are the vertices of the rectangles.

a.) What is the volume of this polyhedron ?

b.) Can this polyhedron turn out to be a regular polyhedron ? If yes, what is the condition for this polyhedron to be regular ?

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21 Prove that the volume V and the lateral area S of a right circular cone satisfy the inequality

$$\left(\frac{6V}{\pi}\right)^2 \leq \left(\frac{2S}{\pi\sqrt{3}}\right)^3$$

When does equality occur?

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22 Let P and P' be two parallelograms with equal area, and let their sidelengths be a, b and a', b'. Assume that $a' \leq a \leq b \leq b'$, and moreover, it is possible to place the segment b' such that it completely lies in the interior of the parallelogram P.

Show that the parallelogram P can be partitioned into four polygons such that these four polygons can be composed again to form the parallelogram P'.

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23 Three faces of a tetrahedron are right triangles, while the fourth is not an obtuse triangle.

(a) Prove that a necessary and sufficient condition for the fourth face to be a right triangle is that at some vertex exactly two angles are right.

(b) Prove that if all the faces are right triangles, then the volume of the tetrahedron equals one -sixth the product of the three smallest edges not belonging to the same face.

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24 There are $n \geq 2$ people at a meeting. Show that there exist two people at the meeting who have the same number of friends among the persons at the meeting. (It is assumed that if A is a friend of B, then B is a friend of A; moreover, nobody is his own friend.)

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25 Prove that

$$\tan 730' = \sqrt{6} + \sqrt{2} - \sqrt{3} - 2.$$

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26 Prove the inequality

a.) $(a_1 + a_2 + ... + a_k)^2 \leq k \left(a_1^2 + a_2^2 + ... + a_k^2\right),$

where $k \geq 1$ is a natural number and $a_1, a_2, ..., a_k$ are arbitrary real numbers.

b.) Using the inequality (1), show that if the real numbers $a_1, a_2, ..., a_n$ satisfy the inequality

$$a_1 + a_2 + ... + a_n \geq \sqrt{(n-1)\left(a_1^2 + a_2^2 + ... + a_n^2\right)},$$

then all of these numbers $a_1, a_2, \ldots, a_n$ are non-negative.

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27 Given a point P lying on a line g, and given a circle K. Construct a circle passing through the point P and touching the circle K and the line g.

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28 In the plane, consider a circle with center S and radius 1. Let ABC be an arbitrary triangle having this circle as its incircle, and assume that $SA \leq SB \leq SC$. Find the locus of

a.) all vertices A of such triangles;

b.) all vertices B of such triangles;

c.) all vertices C of such triangles.

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29 A given natural number N is being decomposed in a sum of some consecutive integers.

a.) Find all such decompositions for $N = 500$.

b.) How many such decompositions does the number $N = 2^{\alpha}3^{\beta}5^{\gamma}$ (where α, β and γ are natural numbers) have? Which of these decompositions contain natural summands only?

c.) Determine the number of such decompositions (= decompositions in a sum of consecutive integers; these integers are not necessarily natural) for an arbitrary natural N.

Note by Darij: The 0 is not considered as a natural number.

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30 Let n be a positive integer, prove that :

(a) $\log_{10}(n+1) > \frac{3}{10n} + \log_{10} n$;

(b) $\log n! > \frac{3n}{10}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - 1\right)$.

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31 Solve the equation $|x^2 - 1| + |x^2 - 4| = mx$ as a function of the parameter m. Which pairs (x, m) of integers satisfy this equation?

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32 The side lengths a, b, c of a triangle ABC form an arithmetical progression (such that $b - a = c - b$). The side lengths a_1, b_1, c_1 of a triangle $A_1B_1C_1$ also form an arithmetical progression (with $b_1 - a_1 = c_1 - b_1$). [Hereby, $a = BC$, $b = CA$, $c = AB$, $a_1 = B_1C_1$, $b_1 = C_1A_1$, $c_1 = A_1B_1$.] Moreover, we know that $\measuredangle CAB = \measuredangle C_1A_1B_1$.

Show that triangles ABC and $A_1B_1C_1$ are similar.

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33 Given two internally tangent circles; in the bigger one we inscribe an equilateral triangle. From each of the vertices of this triangle, we draw a tangent to the smaller circle. Prove that the length of one of these tangents equals the sum of the lengths of the two other tangents.

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34 Find all pairs of positive integers $(x;\ y)$ satisfying the equation $2^x = 3^y + 5$.

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35 Let $ax^3 + bx^2 + cx + d$ be a polynomial with integer coefficients a, b, c, d such that ad is an odd number and bc is an even number. Prove that (at least) one root of the polynomial is irrational.

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36 Let $ABCD$ be a quadrilateral inscribed in a circle. Show that the centroids of triangles ABC, CDA, BCD, DAB lie on one circle.

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37 Show that the four perpendiculars dropped from the midpoints of the sides of a cyclic quadrilateral to the respective opposite sides are concurrent.

Note by Darij: A *cyclic quadrilateral* is a quadrilateral inscribed in a circle.

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38 Two concentric circles have radii R and r respectively. Determine the greatest possible number of circles that are tangent to both these circles and mutually nonintersecting. Prove that this number lies between $\frac{3}{2} \cdot \frac{\sqrt{R}+\sqrt{r}}{\sqrt{R}-\sqrt{r}} - 1$ and $\frac{63}{20} \cdot \frac{R+r}{R-r}$.

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39 Consider a circle with center O and radius R, and let A and B be two points in the plane of this circle.

a.) Draw a chord CD of the circle such that CD is parallel to AB, and the point of the intersection P of the lines AC and BD lies on the circle.

b.) Show that generally, one gets two possible points P (P_1 and P_2) satisfying the condition of the above problem, and compute the distance between these two points, if the lengths $OA = a$, $OB = b$ and $AB = d$ are given.

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40 For a positive real number p, find all real solutions to the equation

$$\sqrt{x^2 + 2px - p^2} - \sqrt{x^2 - 2px - p^2} = 1.$$

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41 Given a regular n-gon $A_1A_2...A_n$ (with $n \geq 3$) in a plane. How many triangles of the kind $A_iA_jA_k$ are obtuse ?

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42 Given a finite sequence of integers a_1, a_2, ..., a_n for $n \geq 2$. Show that there exists a subsequence a_{k_1}, a_{k_2}, ..., a_{k_m}, where $1 \leq k_1 \leq k_2 \leq ... \leq k_m \leq n$, such that the number $a_{k_1}^2 + a_{k_2}^2 + ... + a_{k_m}^2$ is divisible by n.

Note by Darij: Of course, the $1 \leq k_1 \leq k_2 \leq ... \leq k_m \leq n$ should be understood as $1 \leq k_1 < k_2 < ... < k_m \leq n$; else, we could take $m = n$ and $k_1 = k_2 = ... = k_m$, so that the number $a_{k_1}^2 + a_{k_2}^2 + ... + a_{k_m}^2 = n^2 a_{k_1}^2$ will surely be divisible by n.

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43 Given 5 points in a plane, no three of them being collinear. Each two of these 5 points are joined with a segment, and every of these segments is painted either red or blue; assume that there is no triangle whose sides are segments of equal color.

a.) Show that:

(1) Among the four segments originating at any of the 5 points, two are red and two are blue.

(2) The red segments form a closed way passing through all 5 given points. (Similarly for the blue segments.)

b.) Give a plan how to paint the segments either red or blue in order to have the condition (no triangle with equally colored sides) satisfied.

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44 What is the greatest number of balls of radius $1/2$ that can be placed within a rectangular box of size $10 \times 10 \times 1$?

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45 An alphabet consists of n letters. What is the maximal length of a word if we know that any two consecutive letters a, b of the word are different and that the word cannot be reduced to a word of the kind $abab$ with $a \neq b$ by removing letters.

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46 Let a, b, c be reals and

$$f(a,b,c) = \left| \frac{|b-a|}{|ab|} + \frac{b+a}{ab} - \frac{2}{c} \right| + \frac{|b-a|}{|ab|} + \frac{b+a}{ab} + \frac{2}{c}$$

Prove that $f(a,b,c) = 4\max\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\}$.

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47 Consider all segments dividing the area of a triangle ABC in two equal parts. Find the length of the shortest segment among them, if the side lengths $a,$ $b,$ c of triangle ABC are given. How many of these shortest segments exist ?

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48 For which real numbers p does the equation $x^2 + px + 3p = 0$ have integer solutions ?

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49 Two mirror walls are placed to form an angle of measure α. There is a candle inside the angle. How many reflections of the candle can an observer see?

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50 For any quadrilateral with the side lengths $a,$ $b,$ $c,$ d and the area $S,$ prove the inequality$S \leq \frac{a+c}{2} \cdot \frac{b+d}{2}$.

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51 Consider n students with numbers $1, 2, \ldots, n$ standing in the order $1, 2, \ldots, n.$ Upon a command, any of the students either remains on his place or switches his place with another student. (Actually, if student A switches his place with student $B,$ then B cannot switch his place with any other student C any more until the next command comes.)

Is it possible to arrange the students in the order $n, 1, 2, \ldots, n-1$ after two commands ?

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52 A figure with area 1 is cut out of paper. We divide this figure into 10 parts and color them in 10 different colors. Now, we turn around the piece of paper, divide the same figure on the other side of the paper in 10 parts again (in some different way). Show that we can color these new parts in the same 10 colors again (hereby, different parts should have different colors) such that the sum of the areas of all parts of the figure colored with the same color on both sides is $\geq \frac{1}{10}$.

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53 Prove that in every convex hexagon of area S one can draw a diagonal that cuts off a triangle of area not exceeding $\frac{1}{6}S$.

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54 We take 100 consecutive natural numbers a_1, a_2, ..., a_{100}. Determine the last two digits of the number $a_1^8 + a_2^8 + ... + a_{100}^8$.

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55 Given the vertex A and the centroid M of a triangle ABC, find the locus of vertices B such that all the angles of the triangle lie in the interval $[40^\circ, 70^\circ]$.

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56 In a tetrahedron, all three pairs of opposite (skew) edges are mutually perpendicular. Prove that the midpoints of the six edges of the tetrahedron lie on one sphere.

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57 Is it possible to choose a set of 100 (or 200) points on the boundary of a cube such that this set is fixed under each isometry of the cube into itself? Justify your answer.

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58 In a mathematical contest, three problems, A, B, C were posed. Among the participants ther were 25 students who solved at least one problem each. Of all the contestants who did not solve problem A, the number who solved B was twice the number who solved C. The number of students who solved only problem A was one more than the number of students who solved A and at least one other problem. Of all students who solved just one problem, half did not solve problem A. How many students solved only problem B?

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59 Let a, b, c be the lengths of the sides of a triangle, and α, β, γ respectively, the angles opposite these sides. Prove that if

$$a + b = \tan \frac{\gamma}{2}(a \tan \alpha + b \tan \beta)$$

the triangle is isosceles.

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60 Prove that the sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.

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61 Prove that for every natural number n, and for every real number $x \neq \frac{k\pi}{2^t}$ ($t = 0, 1, \dots, n$; k any integer)

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \dots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x$$

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62 Solve the system of equations

$$|a_1 - a_2|x_2 + |a_1 - a_3|x_3 + |a_1 - a_4|x_4 = 1$$

$$|a_2 - a_1|x_1 + |a_2 - a_3|x_3 + |a_2 - a_4|x_4 = 1$$
$$|a_3 - a_1|x_1 + |a_3 - a_2|x_2 + |a_3 - a_4|x_4 = 1$$
$$|a_4 - a_1|x_1 + |a_4 - a_2|x_2 + |a_4 - a_3|x_3 = 1$$

where a_1, a_2, a_3, a_4 are four different real numbers.

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63 Let ABC be a triangle, and let P, Q, R be three points in the interiors of the sides BC, CA, AB of this triangle. Prove that the area of at least one of the three triangles AQR, BRP, CPQ is less than or equal to one quarter of the area of triangle ABC.

Alternative formulation: Let ABC be a triangle, and let P, Q, R be three points on the segments BC, CA, AB, respectively. Prove that
$\min\left\{\left|AQR\right|, \left|BRP\right|, \left|CPQ\right|\right\} \leq \frac14 \cdot \left|ABC\right|$,

where the abbreviation $\left|P_1P_2P_3\right|$ denotes the (non-directed) area of an arbitrary triangle $P_1P_2P_3$.
