IMO Longlists 1967

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– Bulgaria

1 Prove that all numbers of the sequence

$$\frac{107811}{3}, \quad \frac{110778111}{3}, \frac{111077781111}{3}, \quad \ldots$$

are exact cubes.

2 Prove that

$$\frac{1}{3}n^2 + \frac{1}{2}n + \frac{1}{6} \geq (n!)^{\frac{2}{n}},$$

and let $n \geq 1$ be an integer. Prove that this inequality is only possible in the case $n = 1$.

3 Prove the trigonometric inequality $\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{16}$, when $x \in \left(0, \frac{\pi}{2}\right)$.

4 Suppose, medians m_a and m_b of a triangle are orthogonal. Prove that:
(a) The medians of the triangle correspond to the sides of a right-angled triangle.
(b) If a, b, c are the side-lengths of the triangle, then, the following inequality holds:

$$5(a^2 + b^2 - c^2) \geq 8ab$$

5 Solve the system of equations:
$x^2 + x - 1 = y$
$y^2 + y - 1 = z$
$z^2 + z - 1 = x.$

6 Solve the system of equations:
$|x + y| + |1 - x| = 6$
$|x + y + 1| + |1 - y| = 4.$

– Czechoslovakia

7 Find all real solutions of the system of equations:

$$\sum_{k=1}^{n} x_k^i = a^i$$

for $i = 1, 2, \ldots, n.$

8 The parallelogram $ABCD$ has $AB = a, AD = 1, \angle BAD = A$, and the triangle ABD has all angles acute. Prove that circles radius 1 and center A, B, C, D cover the parallelogram if and only

$$a \leq \cos A + \sqrt{3} \sin A.$$

9 Circle k and its diameter AB are given. Find the locus of the centers of circles inscribed in the triangles having one vertex on AB and two other vertices on $k.$

10 The square $ABCD$ has to be decomposed into n triangles (which are not overlapping) and which have all angles acute. Find the smallest integer n for which there exist a solution of that problem and for such n construct at least one decomposition. Answer whether it is possible to ask moreover that (at least) one of these triangles has the perimeter less than an arbitrarily given positive number.

11 Let n be a positive integer. Find the maximal number of non-congruent triangles whose sides lengths are integers $\leq n.$

12 Given a segment AB of the length 1, define the set M of points in the following way: it contains two points A, B, and also all points obtained from A, B by iterating the following rule: With every pair of points X, Y the set M contains also the point Z of the segment XY for which $YZ = 3XZ.$

– Germany, DR

13 Find whether among all quadrilaterals, whose interiors lie inside a semi-circle of radius r, there exist one (or more) with maximum area. If so, determine their shape and area.

14 Which fractions $\dfrac{p}{q}$, where p, q are positive integers < 100, is closest to $\sqrt{2}$? Find all digits after the point in decimal representation of that fraction which coincide with digits in decimal representation of $\sqrt{2}$ (without using any table).

15 Suppose $\tan \alpha = \dfrac{p}{q}$, where p and q are integers and $q \neq 0$. Prove that the number $\tan \beta$ for which $\tan 2\beta = \tan 3\alpha$ is rational only when $p^2 + q^2$ is the square of an integer.

16 Prove the following statement: If r_1 and r_2 are real numbers whose quotient is irrational, then any real number x can be approximated arbitrarily well by the numbers of the form $z_{k_1,k_2} = k_1 r_1 + k_2 r_2$ integers, i.e. for every number x and every positive real number p two integers k_1 and k_2 can be found so that $|x - (k_1 r_1 + k_2 r_2)| < p$ holds.

–	Great Britain
17	Let k, m, n be natural numbers such that $m+k+1$ is a prime greater than $n+1$. Let $c_s = s(s+1)$. Prove that $$(c_{m+1} - c_k)(c_{m+2} - c_k) \dots (c_{m+n} - c_k)$$ is divisible by the product $c_1 c_2 \dots c_n$.
18	If x is a positive rational number show that x can be uniquely expressed in the form $x = \sum_{k=1}^{n} \frac{a_k}{k!}$ where $a_1, a_2, \dots$ are integers, $0 \leq a_n \leq n-1$, for $n > 1$, and the series terminates. Show that x can be expressed as the sum of reciprocals of different integers, each of which is greater than 10^6.
19	The n points $P_1, P_2, \dots, P_n$ are placed inside or on the boundary of a disk of radius 1 in such a way that the minimum distance D_n between any two of these points has its largest possible value D_n. Calculate D_n for $n = 2$ to 7. and justify your answer.
–	Hungary
20	In the space $n \geq 3$ points are given. Every pair of points determines some distance. Suppose all distances are different. Connect every point with the nearest point. Prove that it is impossible to obtain (closed) polygonal line in such a way.
21	Without using tables, find the exact value of the product: $$P = \prod_{k=1}^{7} \cos\left(\frac{k\pi}{15}\right).$$
22	Let k_1 and k_2 be two circles with centers O_1 and O_2 and equal radius r such that $O_1O_2 = r$. Let A and B be two points lying on the circle k_1 and being symmetric to each other with respect to the line O_1O_2. Let P be an arbitrary point on k_2. Prove that $$PA^2 + PB^2 \geq 2r^2.$$
23	Prove that for an arbitrary pair of vectors f and g in the space the inequality $$af^2 + bfg + cg^2 \geq 0$$

holds if and only if the following conditions are fulfilled:

$$a \geq 0, \quad c \geq 0, \quad 4ac \geq b^2.$$

24 In a sports meeting a total of m medals were awarded over n days. On the first day one medal and $\frac{1}{7}$ of the remaining medals were awarded. On the second day two medals and $\frac{1}{7}$ of the remaining medals were awarded, and so on. On the last day, the remaining n medals were awarded. How many medals did the meeting last, and what was the total number of medals ?

25 Three disks of diameter d are touching a sphere in their centers. Besides, every disk touches the other two disks. How to choose the radius R of the sphere in order that axis of the whole figure has an angle of 60° with the line connecting the center of the sphere with the point of the disks which is at the largest distance from the axis ? (The axis of the figure is the line having the property that rotation of the figure of 120° around that line brings the figure in the initial position. Disks are all on one side of the plane, passing through the center of the sphere and orthogonal to the axis).

– Italy

26 Let $ABCD$ be a regular tetrahedron. To an arbitrary point M on one edge, say CD, corresponds the point $P = P(M)$ which is the intersection of two lines AH and BK, drawn from A orthogonally to BM and from B orthogonally to AM. What is the locus of P when M varies ?

27 Which regular polygon can be obtained (and how) by cutting a cube with a plane ?

28 Find values of the parameter u for which the expression

$$y = \frac{\tan(x-u) + \tan(x) + \tan(x+u)}{\tan(x-u)\tan(x)\tan(x+u)}$$

does not depend on x.

29 $A_0B_0C_0$ and $A_1B_1C_1$ are acute-angled triangles. Describe, and prove, how to construct the triangle ABC with the largest possible area which is circumscribed about $A_0B_0C_0$ (so BC contains B_0, CA contains B_0, and AB contains C_0) and similar to $A_1B_1C_1$.

– Mongolia

30 Given $m+n$ numbers: a_i, $i = 1, 2, \ldots, m$, b_j, $j = 1, 2, \ldots, n$, determine the number of pairs (a_i, b_j) for which $|i-j| \geq k$, where k is a non-negative integer.

31 An urn contains balls of k different colors; there are n_i balls of $i-th$ color. Balls are selected at random from the urn, one by one, without replacement, until among the selected balls m balls of the same color appear. Find the greatest number of selections.

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32 Determine the volume of the body obtained by cutting the ball of radius R by the trihedron with vertex in the center of that ball, it its dihedral angles are $\alpha, \beta, \gamma.$

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33 In what case does the system of equations
$x + y + mz = a$
$x + my + z = b$
$mx + y + z = c$
have a solution? Find conditions under which the unique solution of the above system is an arithmetic progression.

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34 Faces of a convex polyhedron are six squares and 8 equilateral triangles and each edge is a common side for one triangle and one square. All dihedral angles obtained from the triangle and square with a common edge, are equal. Prove that it is possible to circumscribe a sphere around the polyhedron, and compute the ratio of the squares of volumes of that polyhedron and of the ball whose boundary is the circumscribed sphere.

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35 Prove the identity

$$\sum_{k=0}^{n} \binom{n}{k} \left(\tan \frac{x}{2}\right)^{2k} \left(1 + \frac{2^k}{\left(1 - \tan^2 \frac{x}{2}\right)^k}\right) = \sec^{2n} \frac{x}{2} + \sec^n x$$

for any natural number n and any angle $x.$

– Poland

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36 Prove this proposition: Center the sphere circumscribed around a tetrahedron which coincides with the center of a sphere inscribed in that tetrahedron if and only if the skew edges of the tetrahedron are equal.

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37 Prove that for arbitrary positive numbers the following inequality holds

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a^8 + b^8 + c^8}{a^3 b^3 c^3}.$$

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38 Does there exist an integer such that its cube is equal to $3n^2 + 3n + 7,$ where n is an integer.

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39 Show that the triangle whose angles satisfy the equality

$$\frac{sin^2(A) + sin^2(B) + sin^2(C)}{cos^2(A) + cos^2(B) + cos^2(C)} = 2$$

is a rectangular triangle.

40 Prove that a tetrahedron with just one edge length greater than 1 has volume at most $\frac{1}{8}$.

41 A line l is drawn through the intersection point H of altitudes of acute-angle triangles. Prove that symmetric images l_a, l_b, l_c of l with respect to the sides BC, CA, AB have one point in common, which lies on the circumcircle of ABC.

– Romania

42 Decompose the expression into real factors:

$$E = 1 - \sin^5(x) - \cos^5(x).$$

43 The equation

$$x^5 + 5\lambda x^4 - x^3 + (\lambda\alpha - 4)x^2 - (8\lambda + 3)x + \lambda\alpha - 2 = 0$$

is given. Determine α so that the given equation has exactly (i) one root or (ii) two roots, respectively, independent from λ.

44 Suppose that p and q are two different positive integers and x is a real number. Form the product $(x+p)(x+q)$. Find the sum $S(x,n) = \sum(x+p)(x+q)$, where p and q take values from 1 to n. Does there exist integer values of x for which $S(x,n) = 0$.

45 **(i)** Solve the equation:

$$\sin^3(x) + \sin^3\left(\frac{2\pi}{3} + x\right) + \sin^3\left(\frac{4\pi}{3} + x\right) + \frac{3}{4}\cos 2x = 0.$$

(ii) Supposing the solutions are in the form of arcs AB with one end at the point A, the beginning of the arcs of the trigonometric circle, and P a regular polygon inscribed in the circle with one vertex in A, find:

1) The subsets of arcs having the other end in B in one of the vertices of the regular dodecagon.
2) Prove that no solution can have the end B in one of the vertices of polygon P whose number of sides is prime or having factors other than 2 or 3.

46 If x, y, z are real numbers satisfying relations

$$x + y + z = 1 \quad \text{and} \quad \arctan x + \arctan y + \arctan z = \frac{\pi}{4},$$

prove that $x^{2n+1} + y^{2n+1} + z^{2n+1} = 1$ holds for all positive integers n.

47 Prove the following inequality:

$$\prod_{i=1}^{k} x_i \cdot \sum_{i=1}^{k} x_i^{n-1} \leq \sum_{i=1}^{k} x_i^{n+k-1},$$

where $x_i > 0$, $k \in \mathbb{N}, n \in \mathbb{N}$.

– Sweden

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48 Determine all positive roots of the equation $x^x = \frac{1}{\sqrt{2}}$.

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49 Let n and k be positive integers such that $1 \leq n \leq N+1$, $1 \leq k \leq N+1$. Show that:

$$\min_{n \neq k} |\sin n - \sin k| < \frac{2}{N}.$$

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50 The function $\varphi(x,y,z)$ defined for all triples (x,y,z) of real numbers, is such that there are two functions f and g defined for all pairs of real numbers, such that

$$\varphi(x,y,z) = f(x+y,z) = g(x,y+z)$$

for all real numbers x, y and z. Show that there is a function h of one real variable, such that

$$\varphi(x,y,z) = h(x+y+z)$$

for all real numbers x, y and z.

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51 A subset S of the set of integers 0 - 99 is said to have property A if it is impossible to fill a crossword-puzzle with 2 rows and 2 columns with numbers in S (0 is written as 00, 1 as 01, and so on). Determine the maximal number of elements in the set S with the property A.

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52 In the plane a point O is and a sequence of points $P_1, P_2, P_3, \ldots$ are given. The distances $OP_1, OP_2, OP_3, \ldots$ are $r_1, r_2, r_3, \ldots$ Let α satisfies $0 < \alpha < 1$. Suppose that for every n the distance from the point P_n to any other point of the sequence is $\geq r_n^{\alpha}$. Determine the exponent β, as large as possible such that for some C independent of n

$$r_n \geq Cn^{\beta}, n = 1, 2, \ldots$$

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53 In making Euclidean constructions in geometry it is permitted to use a ruler and a pair of compasses. In the constructions considered in this question no compasses are permitted, but the ruler is assumed to have two parallel edges, which can be used for constructing two parallel lines through two given points whose distance is at least equal to the breadth of the rule. Then the distance between the parallel lines is equal to the breadth of the ruler. Carry through the following constructions with such a ruler. Construct:

a) The bisector of a given angle.
b) The midpoint of a given rectilinear line segment.
c) The center of a circle through three given non-collinear points.
d) A line through a given point parallel to a given line.

– Soviet Union

54 Is it possible to find a set of 100 (or 200) points on the boundary of a cube such that this set remains fixed under all rotations which leave the cube fixed ?

55 Find all x for which, for all n,

$$\sum_{k=1}^{n} \sin kx \leq \frac{\sqrt{3}}{2}.$$

56 In a group of interpreters each one speaks one of several foreign languages, 24 of them speak Japanese, 24 Malaysian, 24 Farsi. Prove that it is possible to select a subgroup in which exactly 12 interpreters speak Japanese, exactly 12 speak Malaysian and exactly 12 speak Farsi.

57 Let $a_1, \ldots, a_8$ be reals, not all equal to zero. Let

$$c_n = \sum_{k=1}^{8} a_k^n$$

for $n = 1, 2, 3, \ldots$. Given that among the numbers of the sequence (c_n), there are infinitely many equal to zero, determine all the values of n for which $c_n = 0$.

58 A linear binomial $l(z) = Az + B$ with complex coefficients A and B is given. It is known that the maximal value of $|l(z)|$ on the segment $-1 \leq x \leq 1$ $(y = 0)$ of the real line in the complex plane $z = x + iy$ is equal to M. Prove that for every z

$$|l(z)| \leq M\rho,$$

where ρ is the sum of distances from the point $P = z$ to the points $Q_1 : z = 1$ and $Q_3 : z = -1$.

59 On the circle with center 0 and radius 1 the point A_0 is fixed and points $A_1, A_2, \ldots, A_{999}, A_{1000}$ are distributed in such a way that the angle $\angle A_0 0 A_k = k$ (in radians). Cut the circle at points $A_0, A_1, \ldots, A_{1000}$. How many arcs with different lengths are obtained. ?