IMO Longlists 1969

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1 $(BEL1)$ A parabola P_1 with equation $x^2 - 2py = 0$ and parabola P_2 with equation $x^2 + 2py = 0, p > 0$, are given. A line t is tangent to P_2. Find the locus of pole M of the line t with respect to P_1.

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2 $(BEL2)(a)$ Find the equations of regular hyperbolas passing through the points $A(\alpha, 0), B(\beta, 0)$, and $C(0, \gamma)$. (b) Prove that all such hyperbolas pass through the orthocenter H of the triangle ABC. (c) Find the locus of the centers of these hyperbolas. (d) Check whether this locus coincides with the nine-point circle of the triangle ABC.

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3 $(BEL3)$ Construct the circle that is tangent to three given circles.

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4 $(BEL4)$ Let O be a point on a nondegenerate conic. A right angle with vertex O intersects the conic at points A and B. Prove that the line AB passes through a fixed point located on the normal to the conic through the point O.

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5 $(BEL5)$ Let G be the centroid of the triangle OAB. (a) Prove that all conics passing through the points O, A, B, G are hyperbolas. (b) Find the locus of the centers of these hyperbolas.

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6 $(BEL6)$ Evaluate $\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)^{10}$ in two different ways and prove that $\binom{10}{1} - \binom{10}{3} + \frac{1}{2}\binom{10}{5} = 2^4$

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7 $(BUL1)$ Prove that the equation $\sqrt{x^3 + y^3 + z^3} = 1969$ has no integral solutions.

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8 Find all functions f defined for all x that satisfy the condition $xf(y) + yf(x) = (x + y)f(x)f(y)$, for all x and y. Prove that exactly two of them are continuous.

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9 $(BUL3)$ One hundred convex polygons are placed on a square with edge of length $38cm$. The area of each of the polygons is smaller than πcm^2, and the perimeter of each of the polygons is smaller than $2\pi cm$. Prove that there exists a disk with radius 1 in the square that does not intersect any of the polygons.

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10 $(BUL4)$ Let M be the point inside the right-angled triangle $ABC (\angle C = 90^\circ)$ such that $\angle MAB = \angle MBC = \angle MCA = \phi$. Let Ψ be the acute angle between the medians of AC and BC. Prove that $\frac{\sin(\phi+\Psi)}{\sin(\phi-\Psi)} = 5$.

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11 $(BUL5)$ Let Z be a set of points in the plane. Suppose that there exists a pair of points that cannot be joined by a polygonal line not passing through any point of Z. Let us call such a pair of points unjoinable. Prove that for each real $r > 0$ there exists an unjoinable pair of points separated by distance r.

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12 $(CZS1)$ Given a unit cube, find the locus of the centroids of all tetrahedra whose vertices lie on the sides of the cube.

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13 $(CZS2)$ Let p be a prime odd number. Is it possible to find $p-1$ natural numbers $n+1, n+2, ..., n+p-1$ such that the sum of the squares of these numbers is divisible by the sum of these numbers?

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14 $(CZS3)$ Let a and b be two positive real numbers. If x is a real solution of the equation $x^2+px+q=0$ with real coefficients p and q such that $|p| \le a, |q| \le b$, prove that $|x| \le \frac{1}{2}(a+\sqrt{a^2+4b})$ Conversely, if x satisfies the above inequality, prove that there exist real numbers p and q with $|p| \le a, |q| \le b$ such that x is one of the roots of the equation $x^2+px+q=0$.

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15 $(CZS4)$ Let $K_1, \cdots, K_n$ be nonnegative integers. Prove that $K_1!K_2!\cdots K_n! \ge \left[\frac{K}{n}\right]!^n$, where $K = K_1 + \cdots + K_n$

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16 $(CZS5)$ A convex quadrilateral $ABCD$ with sides $AB = a, BC = b, CD = c, DA = d$ and angles $\alpha = \angle DAB, \beta = \angle ABC, \gamma = \angle BCD$, and $\delta = \angle CDA$ is given. Let $s = \frac{a+b+c+d}{2}$ and P be the area of the quadrilateral. Prove that $P^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\frac{\alpha+\gamma}{2}$

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17 $(CZS6)$ Let d and p be two real numbers. Find the first term of an arithmetic progression $a_1, a_2, a_3, \cdots$ with difference d such that $a_1a_2a_3a_4 = p$. Find the number of solutions in terms of d and p.

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18 $(FRA1)$ Let a and b be two nonnegative integers. Denote by $H(a,b)$ the set of numbers n of the form $n = pa + qb$, where p and q are positive integers. Determine $H(a) = H(a,a)$. Prove that if $a \neq b$, it is enough to know all the sets $H(a,b)$ for coprime numbers a, b in order to know all the sets $H(a,b)$. Prove that in the case of coprime numbers a and $b, H(a,b)$ contains all numbers greater than or equal to $\omega = (a-1)(b-1)$ and also $\frac{\omega}{2}$ numbers smaller than ω

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19 $(FRA2)$ Let n be an integer that is not divisible by any square greater than 1. Denote by x_m the last digit of the number x^m in the number system with base n. For which integers x is it possible for x_m to be 0? Prove that the sequence x_m is periodic with period t independent of x. For which x do we have $x_t = 1$. Prove that if m and x are relatively prime, then $0_m, 1_m, ..., (n-1)_m$ are different numbers. Find the minimal period t in terms of n. If n does not meet the given condition, prove that it is possible to have $x_m = 0 \neq x_1$ and that the sequence is periodic starting only from some number $k > 1$.

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20 $(FRA3)$ A polygon (not necessarily convex) with vertices in the lattice points of a rectangular grid is given. The area of the polygon is S. If I is the number of lattice points that are strictly in the interior of the polygon and B the number of lattice points on the border of the polygon, find the number $T = 2S - B - 2I + 2$.

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21 $(FRA4)$ A right-angled triangle OAB has its right angle at the point B. An arbitrary circle with center on the line OB is tangent to the line OA. Let AT be the tangent to the circle different from OA (T is the point of tangency). Prove that the median from B of the triangle OAB intersects AT at a point M such that $MB = MT$.

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22 $(FRA5)$ Let $\alpha(n)$ be the number of pairs (x, y) of integers such that $x + y = n, 0 \leq y \leq x$, and let $\beta(n)$ be the number of triples (x, y, z) such that$x + y + z = n$ and $0 \leq z \leq y \leq x$. Find a simple relation between $\alpha(n)$ and the integer part of the number $\frac{n+2}{2}$ and the relation among $\beta(n), \beta(n-3)$ and $\alpha(n)$. Then evaluate $\beta(n)$ as a function of the residue of n modulo 6. What can be said about $\beta(n)$ and $1 + \frac{n(n+6)}{12}$? And what about $\frac{(n+3)^2}{6}$?
Find the number of triples (x, y, z) with the property $x + y + z \leq n, 0 \leq z \leq y \leq x$ as a function of the residue of n modulo 6.What can be said about the relation between this number and the number $\frac{(n+6)(2n^2+9n+12)}{72}$?

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23 $(FRA6)$ Consider the integer $d = \frac{a^b-1}{c}$, where a, b, and c are positive integers and $c \leq a$. Prove that the set G of integers that are between 1 and d and relatively prime to d (the number of such integers is denoted by $\phi(d)$) can be partitioned into n subsets, each of which consists of b elements. What can be said about the rational number $\frac{\phi(d)}{b}$?

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24 $(GBR1)$ The polynomial $P(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k$, where $a_0, \cdots, a_k$ are integers, is said to be divisible by an integer m if $P(x)$ is a multiple of m for every integral value of x. Show that if $P(x)$ is divisible by m, then $a_0 \cdot k!$ is a multiple of m. Also prove that if a, k, m are positive integers such that $ak!$ is a multiple of m, then a polynomial $P(x)$ with leading term ax^kcan be found that is divisible by m.

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25 $(GBR2)$ Let a, b, x, y be positive integers such that a and b have no common divisor greater than 1. Prove that the largest number not expressible in the form $ax + by$ is $ab - a - b$. If $N(k)$ is the largest number not expressible in the form $ax + by$ in only k ways, find $N(k)$.

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26 $(GBR3)$ A smooth solid consists of a right circular cylinder of height h and base-radius r, surmounted by a hemisphere of radius r and center O. The solid stands on a horizontal table. One end of a string is attached to a point on the base. The string is stretched (initially being kept in the vertical plane) over the highest point of the solid and held down at the point P on the hemisphere such that OP makes an angle α with the horizontal. Show that if α is small enough, the string will slacken if slightly displaced and no longer remain in a vertical plane. If then pulled tight through P, show that it will cross the common circular section of the hemisphere and cylinder at a point Q such that $\angle SOQ = \phi$, S being where it initially crossed this section,

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and $\sin\phi = \frac{r\tan\alpha}{h}$.

27 $(GBR4)$ The segment AB perpendicularly bisects CD at X. Show that, subject to restrictions, there is a right circular cone whose axis passes through X and on whose surface lie the points A, B, C, D. What are the restrictions?

28 $(GBR5)$ Let us define $u_0 = 0, u_1 = 1$ and for $n \geq 0, u_{n+2} = au_{n+1} + bu_n, a$ and b being positive integers. Express u_n as a polynomial in a and b. Prove the result. Given that b is prime, prove that b divides $a(u_b - 1)$.

29 $(GDR1)$ Find all real numbers λ such that the equation $\sin^4 x - \cos^4 x = \lambda(\tan^4 x - \cot^4 x)$ (a) has no solution, (b) has exactly one solution, (c) has exactly two solutions, (d) has more than two solutions (in the interval $(0, \frac{\pi}{4})$.

30 $(GDR2)^{IMO1}$ Prove that there exist infinitely many natural numbers a with the following property: The number $z = n^4 + a$ is not prime for any natural number n.

31 $(GDR3)$ Find the number of permutations $a_1, \cdots, a_n$ of the set $\{1, 2, ..., n\}$ such that $|a_i - a_{i+1}| \neq 1$ for all $i = 1, 2, ..., n - 1$. Find a recurrence formula and evaluate the number of such permutations for $n \leq 6$.

32 $(GDR4)$ Find the maximal number of regions into which a sphere can be partitioned by n circles.

33 $(GDR5)$ Given a ring G in the plane bounded by two concentric circles with radii R and $\frac{R}{2}$, prove that we can cover this region with 8 disks of radius $\frac{2R}{5}$. (A region is covered if each of its points is inside or on the border of some disk.)

34 $(HUN1)$ Let a and b be arbitrary integers. Prove that if k is an integer not divisible by 3, then $(a + b)^{2k} + a^{2k} + b^{2k}$ is divisible by $a^2 + ab + b^2$

35 $(HUN2)$ Prove that $1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{5}{4}$

36 $(HUN3)$ In the plane 4000 points are given such that each line passes through at most 2 of these points. Prove that there exist 1000 disjoint quadrilaterals in the plane with vertices at these points.

37 $(HUN4)$IMO2 If $a_1, a_2, ..., a_n$ are real constants, and if $y = \cos(a_1 + x) + 2\cos(a_2 + x) + \cdots + n\cos(a_n + x)$ has two zeros x_1 and x_2 whose difference is not a multiple of π, prove that $y = 0$.

38 $(HUN5)$ Let r and $m (r \leq m)$ be natural numbers and $Ak = \frac{2k-1}{2m}\pi$. Evaluate $\frac{1}{m^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\sin(rA_k)\sin(rA_l)\cos$ $rA_l)$

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39 $(HUN6)$ Find the positions of three points A, B, C on the boundary of a unit cube such that $min\{AB, AC, BC\}$ is the greatest possible.

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40 $(MON1)$ Find the number of five-digit numbers with the following properties: there are two pairs of digits such that digits from each pair are equal and are next to each other, digits from different pairs are different, and the remaining digit (which does not belong to any of the pairs) is different from the other digits.

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41 $(MON2)$ Given reals x_0, x_1, α, β, find an expression for the solution of the system

$$x_{n+2} - \alpha x_{n+1} - \beta x_n = 0, \qquad n = 0, 1, 2, \dots$$

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42 $(MON3)$ Let $A_k (1 \leq k \leq h)$ be $n-$element sets such that each two of them have a nonempty intersection. Let A be the union of all the sets A_k, and let B be a subset of A such that for each $k (1 \leq k \leq h)$ the intersection of A_k and B consists of exactly two different elements a_k and b_k. Find all subsets X of the set A with r elements satisfying the condition that for at least one index k, both elements a_k and b_k belong to X.

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43 $(MON4)$ Let p and q be two prime numbers greater than 3. Prove that if their difference is 2^n, then for any two integers m and n, the number $S = p^{2m+1} + q^{2m+1}$ is divisible by 3.

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44 $(MON5)$ Find the radius of the circle circumscribed about the isosceles triangle whose sides are the solutions of the equation $x^2 - ax + b = 0$.

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45 Given $n > 4$ points in the plane, no three collinear. Prove that there are at least $\frac{(n-3)(n-4)}{2}$ convex quadrilaterals with vertices amongst the n points.

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46 $(NET1)$ The vertices of an $(n+1)-$gon are placed on the edges of a regular $n-$gon so that the perimeter of the $n-$gon is divided into equal parts. How does one choose these $n+1$ points in order to obtain the $(n+1)-$gon with (a) maximal area; (b) minimal area?

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47 C is a point on the semicircle diameter AB, between A and B. D is the foot of the perpendicular from C to AB. The circle K_1 is the incircle of ABC, the circle K_2 touches CD, DA and the semicircle, the circle K_3 touches CD, DB and the semicircle. Prove that K_1, K_2 and K_3 have another common tangent apart from AB.

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48 $(NET3)$ Let $x_1, x_2, x_3, x_4,$ and x_5 be positive integers satisfying

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1000,$$

$$x_1 - x_2 + x_3 - x_4 + x_5 > 0,$$

$$x_1 + x_2 - x_3 + x_4 - x_5 > 0,$$

$$-x_1 + x_2 + x_3 - x_4 + x_5 > 0,$$

$$x_1 - x_2 + x_3 + x_4 - x_5 > 0,$$

$$-x_1 + x_2 - x_3 + x_4 + x_5 > 0$$

(a) Find the maximum of $(x_1+x_3)^{x_2+x_4}$ (b) In how many different ways can we choose $x_1, ..., x_5$ to obtain the desired maximum?

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49 $(NET4)$ A boy has a set of trains and pieces of railroad track. Each piece is a quarter of circle, and by concatenating these pieces, the boy obtained a closed railway. The railway does not intersect itself. In passing through this railway, the train sometimes goes in the clockwise direction, and sometimes in the opposite direction. Prove that the train passes an even number of times through the pieces in the clockwise direction and an even number of times in the counterclockwise direction. Also, prove that the number of pieces is divisible by 4.

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50 $(NET5)$ The bisectors of the exterior angles of a pentagon $B_1B_2B_3B_4B_5$ form another pentagon $A_1A_2A_3A_4A_5$. Construct $B_1B_2B_3B_4B_5$ from the given pentagon $A_1A_2A_3A_4A_5$.

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51 $(NET6)$ A curve determined by $y = \sqrt{x^2 - 10x + 52}, 0 \leq x \leq 100,$ is constructed in a rectangular grid. Determine the number of squares cut by the curve.

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52 Prove that a regular polygon with an odd number of edges cannot be partitioned into four pieces with equal areas by two lines that pass through the center of polygon.

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53 $(POL2)$ Given two segments AB and CD not in the same plane, find the locus of points M such that $MA^2 + MB^2 = MC^2 + MD^2$.

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54 $(POL3)$ Given a polynomial $f(x)$ with integer coefficients whose value is divisible by 3 for three integers $k, k+1,$ and $k+2$. Prove that $f(m)$ is divisible by 3 for all integers m.

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55 For each of $k = 1, 2, 3, 4, 5$ find necessary and sufficient conditions on $a > 0$ such that there exists a tetrahedron with k edges length a and the remainder length 1.

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56 Let a and b be two natural numbers that have an equal number n of digits in their decimal expansions. The first m digits (from left to right) of the numbers a and b are equal. Prove that if $m > \frac{n}{2}$, then $a^{\frac{1}{n}} - b^{\frac{1}{n}} < \frac{1}{n}$

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57 Given triangle ABC with points M and N are in the sides AB and AC respectively.
If $\dfrac{BM}{MA}+\dfrac{CN}{NA}=1$, then prove that the centroid of ABC lies on MN .

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58 $(SWE1)$ Six points $P_1,...,P_6$ are given in $3-$dimensional space such that no four of them lie in the same plane. Each of the line segments P_jP_k is colored black or white. Prove that there exists one triangle $P_jP_kP_l$ whose edges are of the same color.

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59 $(SWE2)$ For each $\lambda (0<\lambda<1$ and $\lambda=\frac{1}{n}$ for all $n=1,2,3,\cdots)$, construct a continuous function f such that there do not exist x,y with $0<\lambda<y=x+\lambda\le 1$ for which $f(x)=f(y)$.

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60 $(SWE3)$ Find the natural number n with the following properties: (1) Let $S=\{P_1,P_2,\cdots\}$ be an arbitrary finite set of points in the plane, and r_j the distance from P_j to the origin O. We assign to each P_j the closed disk D_j with center P_j and radius r_j. Then some n of these disks contain all points of S. (2) n is the smallest integer with the above property.

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61 $(SWE4)$ Let $a_0,a_1,a_2,\cdots$ be determined with $a_0=0, a_{n+1}=2a_n+2^n$. Prove that if n is power of 2, then so is a_n

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62 Which natural numbers can be expressed as the difference of squares of two integers?

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63 $(SWE6)$ Prove that there are infinitely many positive integers that cannot be expressed as the sum of squares of three positive integers.

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64 $(USS1)$ Prove that for a natural number $n > 2, (n!)! > n[(n - 1)!]^{n!}$.

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65 $(USS2)$ Prove that for $a>b^2$, the identity $\sqrt{a-b\sqrt{a+b\sqrt{a-b\sqrt{a+\cdots}}}}=\sqrt{a-\frac{3}{4}b^2}-\frac{1}{2}b$

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66 $(USS3)$ (a) Prove that if $0\le a_0\le a_1\le a_2$, then $(a_0+a_1x-a_2x^2)^2\le (a_0+a_1+a_2)^2\left(1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{2}x^3+x^4\right)$
(b) Formulate and prove the analogous result for polynomials of third degree.

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67 Given real numbers x_1,x_2,y_1,y_2,z_1,z_2 satisfying $x_1>0,x_2>0,x_1y_1>z_1^2$, and $x_2y_2>z_2^2$, prove that:

$$\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2}\le\frac{1}{x_1y_1-z_1^2}+\frac{1}{x_2y_2-z_2^2}.$$

Give necessary and sufficient conditions for equality.

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68 $(USS5)$ Given 5 points in the plane, no three of which are collinear, prove that we can choose 4 points among them that form a convex quadrilateral.

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69 $(YUG1)$ Suppose that positive real numbers x_1, x_2, x_3 satisfy $x_1x_2x_3 > 1, x_1 + x_2 + x_3 < \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
Prove that: (a) None of x_1, x_2, x_3 equals 1. (b) Exactly one of these numbers is less than 1.

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70 $(YUG2)$ A park has the shape of a convex pentagon of area $50000\sqrt{3}m^2$. A man standing at an interior point O of the park notices that he stands at a distance of at most $200m$ from each vertex of the pentagon. Prove that he stands at a distance of at least $100m$ from each side of the pentagon.

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71 $(YUG3)$ Let four points $A_i(i = 1, 2, 3, 4)$ in the plane determine four triangles. In each of these triangles we choose the smallest angle. The sum of these angles is denoted by S. What is the exact placement of the points A_i if $S = 180°$?
