IMO Longlists 1970

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by BigSams, Amir Hossein, orl

1 Prove that $\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\le\frac{a+b+c}{2}$, where $a,b,c\in\mathbb{R}^+$.

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2 Prove that the two last digits of 9^{9^9} and $9^{9^{9^9}}$ are the same in decimal representation.

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3 Prove that $(a!\cdot b!)|(a+b)!\ \forall a,b\in\mathbb{N}$.

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4 Solve the system of equations for variables x,y, where $\{a,b\}\in\mathbb{R}$ are constants and $a\neq 0$.

$$x^2+xy=a^2+ab$$

$$y^2+xy=a^2-ab$$

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5 Prove that $\sqrt[n]{\sum_{i=1}^{n}\frac{i}{n+1}}\ge 1$ for $2\le n\in\mathbb{N}$.

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6 There is an equation $\sum_{i=1}^{n}\frac{b_i}{x-a_i}=c$ in x, where all $b_i>0$ and $\{a_i\}$ is a strictly increasing sequence. Prove that it has $n-1$ roots such that $x_{n-1}\le a_n$, and $a_i\le x_i$ for each $i\in\mathbb{N},1\le i\le n-1$.

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7 Let $ABCD$ be an arbitrary quadrilateral. Squares with centers M_1,M_2,M_3,M_4 are constructed on AB,BC,CD,DA respectively, all outwards or all inwards. Prove that $M_1M_3=M_2M_4$ and $M_1M_3\perp M_2M_4$.

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8 Consider a regular $2n$-gon and the n diagonals of it that pass through its center. Let P be a point of the inscribed circle and let $a_1,a_2,\ldots,a_n$ be the angles in which the diagonals mentioned are visible from the point P. Prove that

$$\sum_{i=1}^{n}\tan^2 a_i=2n\frac{\cos^2\frac{\pi}{2n}}{\sin^4\frac{\pi}{2n}}.$$

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9 For even n, prove that $\sum_{i=1}^{n}\left((-1)^{i+1}\cdot\frac{1}{i}\right)=2\sum_{i=1}^{n/2}\frac{1}{n+2i}$.

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10 In $\triangle ABC$, prove that $1<\sum_{cyc}\cos A\le\frac{3}{2}$.

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11 Let $ABCD$ and $A'B'C'D'$ be two arbitrary squares in the plane that are oriented in the same direction. Prove that the quadrilateral formed by the midpoints of AA', BB', CC', DD' is a square.

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12 Let $\{x_i\}, 1 \leq i \leq 6$ be a given set of six integers, none of which are divisible by 7. (a) Prove that at least one of the expressions of the form $x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 \pm x_6$ is divisible by 7, where the $\pm$ signs are independent of each other. (b) Generalize the result to every prime number.

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13 Each side of an arbitrary $\triangle ABC$ is divided into equal parts, and lines parallel to AB, BC, CA are drawn through each of these points, thus cutting $\triangle ABC$ into small triangles. Points are assigned a number in the following manner. (1) A, B, C are assigned $1, 2, 3$ respectively (2) Points on AB are assigned 1 or 2 (3) Points on BC are assigned 2 or 3 (4) Points on CA are assigned 3 or 1
Prove that there must exist a small triangle whose vertices are marked by $1, 2, 3$.

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14 Let $\alpha + \beta + \gamma = \pi$. Prove that $\sum_{cyc} \sin 2\alpha = 2 \cdot \left(\sum_{cyc} \sin \alpha\right) \cdot \left(\sum_{cyc} \cos \alpha\right) - 2 \sum_{cyc} \sin \alpha$.

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15 Given $\triangle ABC$, let R be its circumradius and q be the perimeter of its excentral triangle. Prove that $q \leq 6\sqrt{3}R$.
Typesetter's Note: the excentral triangle has vertices which are the excenters of the original triangle.

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16 Show that the equation $\sqrt{2 - x^2} + \sqrt[3]{3 - x^3} = 0$ has no real roots.

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17 In the tetrahedron $ABCD, \angle BDC = 90^o$ and the foot of the perpendicular from D to ABC is the intersection of the altitudes of ABC. Prove that:

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

When do we have equality?

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18 Find all positive integers n such that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two subsets so that the product of the numbers in each subset is equal.

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19 Let $1 < n \in \mathbb{N}$ and $1 \leq a \in \mathbb{R}$ and there are n number of $x_i, i \in \mathbb{N}, 1 \leq i \leq n$ such that $x_1 = 1$ and $\frac{x_i}{x_{i-1}} = a + \alpha_i$ for $2 \leq i \leq n$, where $\alpha_i \leq \frac{1}{i(i+1)}$. Prove that $\sqrt[n-1]{x_n} < a + \frac{1}{n-1}$.

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20 Let M be an interior point of the tetrahedron $ABCD$. Prove that

$$\overrightarrow{MA}\,\text{vol}(MBCD) + \overrightarrow{MB}\,\text{vol}(MACD) + \overrightarrow{MC}\,\text{vol}(MABD) + \overrightarrow{MD}\,\text{vol}(MABC) = 0$$

(vol$(PQRS)$ denotes the volume of the tetrahedron $PQRS$).

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21 In the triangle ABC let B' and C' be the midpoints of the sides AC and AB respectively and H the foot of the altitude passing through the vertex A. Prove that the circumcircles of the triangles $AB'C'$,$BC'H$, and $B'CH$ have a common point I and that the line HI passes through the midpoint of the segment $B'C'$.

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23 Let E be a finite set, P_E the family of its subsets, and f a mapping from P_E to the set of non-negative reals, such that for any two disjoint subsets A, B of E, $f(A \cup B) = f(A) + f(B)$. Prove that there exists a subset F of E such that if with each $A \subset E$, we associate a subset A' consisting of elements of A that are not in F, then $f(A) = f(A')$ and $f(A)$ is zero if and only if A is a subset of F.

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24 Let $\{n, p\} \in \mathbb{N} \cup \{0\}$ such that $2p \leq n$. Prove that $\frac{(n-p)!}{p!} \leq \left(\frac{n+1}{2}\right)^{n-2p}$. Determine all conditions under which equality holds.

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25 A real function f is defined for $0 \leq x \leq 1$, with its first derivative f' defined for $0 \leq x \leq 1$ and its second derivative f'' defined for $0 < x < 1$. Prove that if $f(0) = f'(0) = f'(1) = f(1) - 1 = 0$, then there exists a number $0 < y < 1$ such that $|f''(y)| \geq 4$.

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26 Consider a finite set of vectors in space $\{a_1, a_2, ..., a_n\}$ and the set E of all vectors of the form $x = \sum_{i=1}^{n} \lambda_i a_i$, where $\lambda_i \in \mathbb{R}^+ \cup \{0\}$. Let F be the set consisting of all the vectors in E and vectors parallel to a given plane P. Prove that there exists a set of vectors $\{b_1, b_2, ..., b_p\}$ such that F is the set of all vectors y of the form $y = \sum_{i=1}^{p} \mu_i b_i$, where $\mu_i \in \mathbb{R}^+ \cup \{0\}$.

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27 Find a $n \in \mathbb{N}$ such that for all primes p, n is divisible by p if and only if n is divisible by $p - 1$.

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28 A set G with elements $u, v, w...$ is a Group if the following conditions are fulfilled: (i) There is a binary operation $\circ$ defined on G such that $\forall\{u, v\} \in G$ there is a $w \in G$ with $u \circ v = w$. (ii) This operation is associative; i.e. $(u \circ v) \circ w = u \circ (v \circ w)$ $\forall\{u, v, w\} \in G$. (iii) $\forall\{u, v\} \in G$, there exists an element $x \in G$ such that $u \circ x = v$, and an element $y \in G$ such that $y \circ u = v$.

Let K be a set of all real numbers greater than 1. On K is defined an operation by $a \circ b = ab - \sqrt{(a^2 - 1)(b^2 - 1)}$. Prove that K is a Group.

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29 Prove that the equation $4^x + 6^x = 9^x$ has no rational solutions.

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30 Let $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$ be real numbers. Prove that

$$1+\sum_{i=1}^{n}(u_i+v_i)^2 \leq \frac{4}{3}\left(1+\sum_{i=1}^{n}u_i^2\right)\left(1+\sum_{i=1}^{n}v_i^2\right).$$

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31 Prove that for any triangle with sides a, b, c and area P the following inequality holds:

$$P \leq \frac{\sqrt{3}}{4}(abc)^{2/3}.$$

Find all triangles for which equality holds.

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32 Let there be given an acute angle $\angle AOB = 3\alpha$, where $\overline{OA} = \overline{OB}$. The point A is the center of a circle with radius $\overline{OA}$. A line s parallel to OA passes through B. Inside the given angle a variable line t is drawn through O. It meets the circle in O and C and the given line s in D, where $\angle AOC = x$. Starting from an arbitrarily chosen position t_0 of t, the series $t_0, t_1, t_2, \dots$ is determined by defining $\overline{BD_{i+1}} = \overline{OC_i}$ for each i (in which C_i and D_i denote the positions of C and D, corresponding to t_i). Making use of the graphical representations of BD and OC as functions of x, determine the behavior of t_i for $i \to \infty$.

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33 The vertices of a given square are clockwise lettered A, B, C, D. On the side AB is situated a point E such that $AE = AB/3$. Starting from an arbitrarily chosen point P_0 on segment AE and going clockwise around the perimeter of the square, a series of points $P_0, P_1, P_2, \dots$ is marked on the perimeter such that $P_iP_{i+1} = AB/3$ for each i. It will be clear that when P_0 is chosen in A or in E, then some P_i will coincide with P_0. Does this possibly also happen if P_0 is chosen otherwise?

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34 In connection with a convex pentagon $ABCDE$ we consider the set of ten circles, each of which contains three of the vertices of the pentagon on its circumference. Is it possible that none of these circles contains the pentagon? Prove your answer.

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35 Find for every value of n a set of numbers p for which the following statement is true: Any convex n-gon can be divided into p isosceles triangles.

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36 Let x, y, z be non-negative real numbers satisfying

$$x^2+y^2+z^2=5 \quad \text{and} \quad yz+zx+xy=2.$$

Which values can the greatest of the numbers x^2-yz, y^2-xz and z^2-xy have?

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37 Solve the set of simultaneous equations

$$\begin{aligned} v^2+w^2+x^2+y^2 &= 6-2u, \\ u^2+w^2+x^2+y^2 &= 6-2v, \\ u^2+v^2+x^2+y^2 &= 6-2w, \\ u^2+v^2+w^2+y^2 &= 6-2x, \\ u^2+v^2+w^2+x^2 &= 6-2y. \end{aligned}$$

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38 Find the greatest integer A for which in any permutation of the numbers $1, 2, \ldots, 100$ there exist ten consecutive numbers whose sum is at least A.

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39 M is any point on the side AB of the triangle ABC. r, r_1, r_2 are the radii of the circles inscribed in ABC, AMC, BMC. q is the radius of the circle on the opposite side of AB to C, touching the three sides of AB and the extensions of CA and CB. Similarly, q_1 and q_2. Prove that $r_1r_2q = rq_1q_2$.

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40 Let ABC be a triangle with angles α, β, γ commensurable with π. Starting from a point P interior to the triangle, a ball reflects on the sides of ABC, respecting the law of reflection that the angle of incidence is equal to the angle of reflection. Prove that, supposing that the ball never reaches any of the vertices A, B, C, the set of all directions in which the ball will move through time is finite. In other words, its path from the moment 0 to infinity consists of segments parallel to a finite set of lines.

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41 Let a cube of side 1 be given. Prove that there exists a point A on the surface S of the cube such that every point of S can be joined to A by a path on S of length not exceeding 2. Also prove that there is a point of S that cannot be joined with A by a path on S of length less than 2.

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42 We have $0 \leq x_i < b$ for $i = 0, 1, \ldots, n$ and $x_n > 0, x_{n-1} > 0$. If $a > b$, and $x_nx_{n-1}\ldots x_0$ represents the number A base a and B base b, whilst $x_{n-1}x_{n-2}\ldots x_0$ represents the number A' base a and B' base b, prove that $A'B < AB'$.

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43 Prove that the equation

$$x^3 - 3\tan\frac{\pi}{12}x^2 - 3x + \tan\frac{\pi}{12} = 0$$

has one root $x_1 = \tan\frac{\pi}{36}$, and find the other roots.

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44 If a, b, c are side lengths of a triangle, prove that

$$(a+b)(b+c)(c+a) \geq 8(a+b-c)(b+c-a)(c+a-b).$$

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45 Let M be an interior point of tetrahedron $VABC$. Denote by A_1, B_1, C_1 the points of intersection of lines MA, MB, MC with the planes VBC, VCA, VAB, and by A_2, B_2, C_2 the points of intersection of lines VA_1, VB_1, VC_1 with the sides BC, CA, AB.

(a) Prove that the volume of the tetrahedron $VA_2B_2C_2$ does not exceed one-fourth of the volume of $VABC$.

(b) Calculate the volume of the tetrahedron $V_1A_1B_1C_1$ as a function of the volume of $VABC$, where V_1 is the point of intersection of the line VM with the plane ABC, and M is the barycenter of $VABC$.

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46 Given a triangle ABC and a plane π having no common points with the triangle, find a point M such that the triangle determined by the points of intersection of the lines MA, MB, MC with π is congruent to the triangle ABC.

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47 Given a polynomial

$$P(x) = ab(a-c)x^3 + (a^3 - a^2c + 2ab^2 - b^2c + abc)x^2 + (2a^2b + b^2c + a^2c + b^3 - abc)x + ab(b+c),$$

where $a, b, c \neq 0$, prove that $P(x)$ is divisible by

$$Q(x) = abx^2 + (a^2 + b^2)x + ab$$

and conclude that $P(x_0)$ is divisible by $(a+b)^3$ for $x_0 = (a+b+1)^n, n \in \mathbb{N}$.

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48 Let a polynomial $p(x)$ with integer coefficients take the value 5 for five different integer values of x. Prove that $p(x)$ does not take the value 8 for any integer x.

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49 For $n \in \mathbb{N}$, let $f(n)$ be the number of positive integers $k \leq n$ that do not contain the digit 9. Does there exist a positive real number p such that $\frac{f(n)}{n} \geq p$ for all positive integers n?

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50 The area of a triangle is S and the sum of the lengths of its sides is L. Prove that $36S \leq L^2\sqrt{3}$ and give a necessary and sufficient condition for equality.

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51 Let p be a prime number. A rational number x, with $0 < x < 1$, is written in lowest terms. The rational number obtained from x by adding p to both the numerator and the denominator differs from x by $1/p^2$. Determine all rational numbers x with this property.

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52 The real numbers $a_0, a_1, a_2, \dots$ satisfy $1 = a_0 \leq a_1 \leq a_2 \leq \dots . b_1, b_2, b_3, \dots$ are defined by $b_n = \sum_{k=1}^{n} \frac{1 - \frac{a_{k-1}}{a_k}}{\sqrt{a_k}}$.

a.) Prove that $0 \leq b_n < 2$.

b.) Given c satisfying $0 \leq c < 2$, prove that we can find a_n so that $b_n > c$ for all sufficiently large n.

53 A square $ABCD$ is divided into $(n-1)^2$ congruent squares, with sides parallel to the sides of the given square. Consider the grid of all n^2 corners obtained in this manner. Determine all integers n for which it is possible to construct a non-degenerate parabola with its axis parallel to one side of the square and that passes through exactly n points of the grid.

54 Let P, Q, R be polynomials and let $S(x) = P(x^3) + xQ(x^3) + x^2R(x^3)$ be a polynomial of degree n whose roots $x_1, \dots, x_n$ are distinct. Construct with the aid of the polynomials P, Q, R a polynomial T of degree n that has the roots $x_1^3, x_2^3, \dots, x_n^3$.

55 A turtle runs away from an UFO with a speed of $0.2\ m/s$. The UFO flies 5 meters above the ground, with a speed of $20\ m/s$. The UFO's path is a broken line, where after flying in a straight path of length ℓ (in meters) it may turn through for any acute angle α such that $\tan\alpha < \frac{\ell}{1000}$. When the UFO's center approaches within 13 meters of the turtle, it catches the turtle. Prove that for any initial position the UFO can catch the turtle.

56 A square hole of depth h whose base is of length a is given. A dog is tied to the center of the square at the bottom of the hole by a rope of length $L > \sqrt{2a^2 + h^2}$, and walks on the ground around the hole. The edges of the hole are smooth, so that the rope can freely slide along it. Find the shape and area of the territory accessible to the dog (whose size is neglected).

57 Let the numbers $1, 2, \dots, n^2$ be written in the cells of an $n \times n$ square board so that the entries in each column are arranged increasingly. What are the smallest and greatest possible sums of the numbers in the k^{th} row? (k a positive integer, $1 \le k \le n$.)

58 Given 100 coplanar points, no three collinear, prove that at most 70% of the triangles formed by the points have all angles acute.

59 For which digits a do exist integers $n \ge 4$ such that each digit of $\frac{n(n+1)}{2}$ equals a ?