IMO Longlists 1971

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1 The points $S(i,j)$ with integer Cartesian coordinates $0 < i \leq n, 0 < j \leq m, m \leq n$, form a lattice. Find the number of:

(a) rectangles with vertices on the lattice and sides parallel to the coordinate axes;

(b) squares with vertices on the lattice and sides parallel to the coordinate axes;

(c) squares in total, with vertices on the lattice.

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2 Let us denote by $s(n) = \sum_{d|n} d$ the sum of divisors of a positive integer n (1 and n included). If n has at most 5 distinct prime divisors, prove that $s(n) < \frac{77}{16}n$. Also prove that there exists a natural number n for which $s(n) < \frac{76}{16}n$ holds.

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3 Let a, b, c be positive real numbers, $0 < a \leq b \leq c$. Prove that for any positive real numbers x, y, z the following inequality holds:

$$(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right) \leq (x+y+z)^2 \cdot \frac{(a+c)^2}{4ac}.$$

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4 Let $x_n = 2^{2^n} + 1$ and let m be the least common multiple of $x_2, x_3, \ldots, x_{1971}$. Find the last digit of m.

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5 Consider a sequence of polynomials $P_0(x), P_1(x), P_2(x), \ldots, P_n(x), \ldots$, where $P_0(x) = 2, P_1(x) = x$ and for every $n \geq 1$ the following equality holds:

$$P_{n+1}(x) + P_{n-1}(x) = xP_n(x).$$

Prove that there exist three real numbers a, b, c such that for all $n \geq 1$,

$$(x^2-4)[P_n^2(x) - 4] = [aP_{n+1}(x) + bP_n(x) + cP_{n-1}(x)]^2.$$

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6 Let squares be constructed on the sides BC, CA, AB of a triangle ABC, all to the outside of the triangle, and let A_1, B_1, C_1 be their centers. Starting from the triangle $A_1B_1C_1$ one analogously obtains a triangle $A_2B_2C_2$. If S, S_1, S_2 denote the areas of triangles $ABC, A_1B_1C_1, A_2B_2C_2$, respectively, prove that $S = 8S_1 - 4S_2$.

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7 In a triangle ABC, let H be its orthocenter, O its circumcenter, and R its circumradius. Prove that:

(a) $|OH| = R\sqrt{1 - 8\cos\alpha \cdot \cos\beta \cdot \cos\gamma}$ where α, β, γ are angles of the triangle ABC;

(b) $O \equiv H$ if and only if ABC is equilateral.

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8 Prove that for every positive integer m we can find a finite set S of points in the plane, such that given any point A of S, there are exactly m points in S at unit distance from A.

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9 The base of an inclined prism is a triangle ABC. The perpendicular projection of B_1, one of the top vertices, is the midpoint of BC. The dihedral angle between the lateral faces through BC and AB is α, and the lateral edges of the prism make an angle β with the base. If r_1, r_2, r_3 are exradii of a perpendicular section of the prism, assuming that in $ABC, \cos^2 A + \cos^2 B + \cos^2 C = 1, \angle A < \angle B < \angle C$, and $BC = a$, calculate $r_1r_2 + r_1r_3 + r_2r_3$.

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10 In how many different ways can three knights be placed on a chessboard so that the number of squares attacked would be maximal?

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11 Find all positive integers n for which the number $1! + 2! + 3! + \cdots + n!$ is a perfect power of an integer.

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12 A system of n numbers $x_1, x_2, \dots, x_n$ is given such that

$$x_1 = \log_{x_{n-1}} x_n, x_2 = \log_{x_n} x_1, \dots, x_n = \log_{x_{n-2}} x_{n-1}.$$

Prove that $\prod_{k=1}^{n} x_k = 1$.

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13 One Martian, one Venusian, and one Human reside on Pluton. One day they make the following conversation:

Martian : I have spent $1/12$ of my life on Pluton.
Human : I also have.
Venusian : Me too.
Martian : But Venusian and I have spend much more time here than you, Human.
Human : That is true. However, Venusian and I are of the same age.
Venusian : Yes, I have lived 300 Earth years.
Martian : Venusian and I have been on Pluton for the past 13 years.

It is known that Human and Martian together have lived 104 Earth years. Find the ages of Martian, Venusian, and Human.*

[i]*: Note that the numbers in the problem are not necessarily in base 10.[/i]

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14 Note that $8^3 - 7^3 = 169 = 13^2$ and $13 = 2^2 + 3^2$. Prove that if the difference between two consecutive cubes is a square, then it is the square of the sum of two consecutive squares.

15 Let $ABCD$ be a convex quadrilateral whose diagonals intersect at O at an angle θ. Let us set $OA = a, OB = b, OC = c$, and $OD = d, c > a > 0$, and $d > b > 0$.

Show that if there exists a right circular cone with vertex V, with the properties:

(1) its axis passes through O, and

(2) its curved surface passes through A, B, C and D, then

$$OV^2 = \frac{d^2b^2(c+a)^2 - c^2a^2(d+b)^2}{ca(d-b)^2 - db(c-a)^2}.$$

Show also that if $\frac{c+a}{d+b}$ lies between $\frac{ca}{db}$ and $\sqrt{\frac{ca}{db}}$, and $\frac{c-a}{d-b} = \frac{ca}{db}$, then for a suitable choice of θ, a right circular cone exists with properties **(1)** and **(2).**

16 Knowing that the system

$$x + y + z = 3,$$
$$x^3 + y^3 + z^3 = 15,$$
$$x^4 + y^4 + z^4 = 35,$$

has a real solution x, y, z for which $x^2 + y^2 + z^2 < 10$, find the value of $x^5 + y^5 + z^5$ for that solution.

17 We are given two mutually tangent circles in the plane, with radii r_1, r_2. A line intersects these circles in four points, determining three segments of equal length. Find this length as a function of r_1 and r_2 and the condition for the solvability of the problem.

18 Let $a_1, a_2, \dots, a_n$ be positive numbers, $m_g = \sqrt[n]{(a_1a_2\cdots a_n)}$ their geometric mean, and $m_a = \frac{(a_1+a_2+\cdots+a_n)}{n}$ their arithmetic mean. Prove that

$$(1+m_g)^n \le (1+a_1)\cdots(1+a_n) \le (1+m_a)^n.$$

19 In a triangle $P_1P_2P_3$ let P_iQ_i be the altitude from P_i for $i = 1, 2, 3$ (Q_i being the foot of the altitude). The circle with diameter P_iQ_i meets the two corresponding sides at two points different from P_i. Denote the length of the segment whose endpoints are these two points by l_i. Prove that $l_1 = l_2 = l_3$.

20 Let M be the circumcenter of a triangle ABC. The line through M perpendicular to CM meets the lines CA and CB at Q and P, respectively. Prove that

$$\frac{\overline{CP}}{\overline{CM}} \cdot \frac{\overline{CQ}}{\overline{CM}} \cdot \frac{\overline{AB}}{\overline{PQ}} = 2.$$

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21 Let

$$E_n = (a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_n) + (a_2 - a_1)(a_2 - a_3) \dots (a_2 - a_n) + \dots + (a_n - a_1)(a_n - a_2) \dots (a_n - a_{n-1}).$$

Let S_n be the proposition that $E_n \geq 0$ for all real a_i. Prove that S_n is true for $n = 3$ and 5, but for no other $n > 2$.

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22 We are given an $n \times n$ board, where n is an odd number. In each cell of the board either $+1$ or -1 is written. Let a_k and b_k denote them products of numbers in the k^{th} row and in the k^{th} column respectively. Prove that the sum $a_1 + a_2 + \cdots + a_n + b_1 + b_2 + \cdots + b_n$ cannot be equal to zero.

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23 Find all integer solutions of the equation

$$x^2 + y^2 = (x - y)^3.$$

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24 Let $A, B,$ and C denote the angles of a triangle. If $\sin^2 A + \sin^2 B + \sin^2 C = 2$, prove that the triangle is right-angled.

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25 Let $ABC, AA_1A_2, BB_1B_2, CC_1C_2$ be four equilateral triangles in the plane satisfying only that they are all positively oriented (i.e., in the counterclockwise direction). Denote the midpoints of the segments A_2B_1, B_2C_1, C_2A_1 by P, Q, R in this order. Prove that the triangle PQR is equilateral.

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26 An infinite set of rectangles in the Cartesian coordinate plane is given. The vertices of each of these rectangles have coordinates $(0,0), (p,0), (p,q), (0,q)$ for some positive integers p, q. Show that there must exist two among them one of which is entirely contained in the other.

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27 Let $n \geq 2$ be a natural number. Find a way to assign natural numbers to the vertices of a regular $2n$-gon such that the following conditions are satisfied:

(1) only digits 1 and 2 are used;

(2) each number consists of exactly n digits;

(3) different numbers are assigned to different vertices;

(4) the numbers assigned to two neighboring vertices differ at exactly one digit.

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28 All faces of the tetrahedron $ABCD$ are acute-angled. Take a point X in the interior of the segment AB, and similarly Y in BC, Z in CD and T in AD.

a.) If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, then prove none of the closed paths $XYZTX$ has minimal length;

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b.) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then there are infinitely many shortest paths $XYZTX$, each with length $2AC \sin k$, where $2k = \angle BAC + \angle CAD + \angle DAB$.

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29 A rhombus with its incircle is given. At each vertex of the rhombus a circle is constructed that touches the incircle and two edges of the rhombus. These circles have radii r_1, r_2, while the incircle has radius r. Given that r_1 and r_2 are natural numbers and that $r_1 r_2 = r$, find $r_1, r_2,$ and r.

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30 Prove that the system of equations

$$2yz + x - y - z = a, 2xz - x + y - z = a, 2xy - x - y + z = a,$$

a being a parameter, cannot have five distinct solutions. For what values of a does this system have four distinct integer solutions?

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31 Determine whether there exist distinct real numbers a, b, c, t for which:

(i) the equation $ax^2 + btx + c = 0$ has two distinct real roots x_1, x_2,

(ii) the equation $bx^2 + ctx + a = 0$ has two distinct real roots x_2, x_3,

(iii) the equation $cx^2 + atx + b = 0$ has two distinct real roots x_3, x_1.

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32 Two half-lines a and b, with the common endpoint O, make an acute angle α. Let A on a and B on b be points such that $OA = OB$, and let b be the line through A parallel to b. Let β be the circle with centre B and radius BO. We construct a sequence of half-lines $c_1, c_2, c_3, \ldots$, all lying inside the angle α, in the following manner.
(i) c_i is given arbitrarily;
(ii) for every natural number k, the circle β intercepts on c_k a segment that is of the same length as the segment cut on b' by a and c_{k+1}.
Prove that the angle determined by the lines c_k and b has a limit as k tends to infinity and find that limit.

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33 A square $2n \times 2n$ grid is given. Let us consider all possible paths along grid lines, going from the centre of the grid to the border, such that (1) no point of the grid is reached more than once, and (2) each of the squares homothetic to the grid having its centre at the grid centre is passed through only once.
(a) Prove that the number of all such paths is equal to $4\prod_{i=2}^{n}(16i - 9)$.
(b) Find the number of pairs of such paths that divide the grid into two congruent figures.
(c) How many quadruples of such paths are there that divide the grid into four congruent parts?

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34 Let $T_k = k - 1$ for $k = 1, 2, 3, 4$ and

$$T_{2k-1} = T_{2k-2} + 2^{k-2}, T_{2k} = T_{2k-5} + 2^k \qquad (k \geq 3).$$

Show that for all k,

$$1 + T_{2n-1} = \left[\frac{12}{7}2^{n-1}\right] \quad \text{and} \quad 1 + T_{2n} = \left[\frac{17}{7}2^{n-1}\right],$$

where $[x]$ denotes the greatest integer not exceeding x.

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35 Prove that we can find an infinite set of positive integers of the from $2^n - 3$ (where n is a positive integer) every pair of which are relatively prime.

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36 The matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

satisfies the inequality $\sum_{j=1}^{n} |a_{j1}x_1 + \cdots + a_{jn}x_n| \leq M$ for each choice of numbers x_i equal to ± 1. Show that

$$|a_{11} + a_{22} + \cdots + a_{nn}| \leq M.$$

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37 Let S be a circle, and $\alpha = \{A_1, \dots, A_n\}$ a family of open arcs in S. Let $N(\alpha) = n$ denote the number of elements in α. We say that α is a covering of S if $\bigcup_{k=1}^{n} A_k \supset S$.
Let $\alpha = \{A_1, \dots, A_n\}$ and $\beta = \{B_1, \dots, B_m\}$ be two coverings of S.
Show that we can choose from the family of all sets $A_i \cap B_j,\ i = 1, 2, \dots, n,\ j = 1, 2, \dots, m$, a covering γ of S such that $N(\gamma) \leq N(\alpha) + N(\beta)$.

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38 Let A, B, C be three points with integer coordinates in the plane and K a circle with radius R passing through A, B, C. Show that $AB \cdot BC \cdot CA \geq 2R$, and if the centre of K is in the origin of the coordinates, show that $AB \cdot BC \cdot CA \geq 4R$.

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39 Two congruent equilateral triangles ABC and $A'B'C'$ in the plane are given. Show that the midpoints of the segments AA', BB', CC' either are collinear or form an equilateral triangle.

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40 Consider the set of grid points (m, n) in the plane, m, n integers. Let σ be a finite subset and define

$$S(\sigma) = \sum_{(m,n)\in\sigma} (100 - |m| - |n|)$$

Find the maximum of S, taken over the set of all such subsets σ.

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41 Let $L_i,\ i = 1, 2, 3$, be line segments on the sides of an equilateral triangle, one segment on each side, with lengths $l_i,\ i = 1, 2, 3$. By L_i^* we denote the segment of length l_i with its midpoint on the midpoint of the corresponding side of the triangle. Let $M(L)$ be the set of points in the plane whose orthogonal projections on the sides of the triangle are in L_1, L_2, and L_3,

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respectively; $M(L^*)$ is defined correspondingly. Prove that if $l_1 \geq l_2 + l_3$, we have that the area of $M(L)$ is less than or equal to the area of $M(L^*)$.

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42 Show that for nonnegative real numbers a, b and integers $n \geq 2$,

$$\frac{a^n + b^n}{2} \geq \left(\frac{a+b}{2}\right)^n$$

When does equality hold?

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43 Let $A = (a_{ij})$, where $i, j = 1, 2, \dots, n$, be a square matrix with all a_{ij} non-negative integers. For each i, j such that $a_{ij} = 0$, the sum of the elements in the ith row and the jth column is at least n. Prove that the sum of all the elements in the matrix is at least $\frac{n^2}{2}$.

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44 Let m and n denote integers greater than 1, and let $\nu(n)$ be the number of primes less than or equal to n. Show that if the equation $\frac{n}{\nu(n)} = m$ has a solution, then so does the equation $\frac{n}{\nu(n)} = m - 1$.

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45 A broken line $A_1A_2 \dots A_n$ is drawn in a 50×50 square, so that the distance from any point of the square to the broken line is less than 1. Prove that its total length is greater than 1248.

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46 Natural numbers from 1 to 99 (not necessarily distinct) are written on 99 cards. It is given that the sum of the numbers on any subset of cards (including the set of all cards) is not divisible by 100. Show that all the cards contain the same number.

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47 A sequence of real numbers $x_1, x_2, \dots, x_n$ is given such that $x_{i+1} = x_i + \frac{1}{30000}\sqrt{1 - x_i^2},\ i = 1, 2, \dots,$ and $x_1 = 0$. Can n be equal to 50000 if $x_n < 1$?

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48 The diagonals of a convex quadrilateral $ABCD$ intersect at a point O. Find all angles of this quadrilateral if $\measuredangle OBA = 30^\circ, \measuredangle OCB = 45^\circ, \measuredangle ODC = 45^\circ$, and $\measuredangle OAD = 30^\circ$.

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49 Let P_1 be a convex polyhedron with vertices $A_1, A_2, \dots, A_9$. Let P_i be the polyhedron obtained from P_1 by a translation that moves A_1 to A_i. Prove that at least two of the polyhedra $P_1, P_2, \dots, P_9$ have an interior point in common.

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51 Suppose that the sides AB and DC of a convex quadrilateral $ABCD$ are not parallel. On the sides BC and AD, pairs of points (M, N) and (K, L) are chosen such that $BM = MN = NC$ and $AK = KL = LD$. Prove that the areas of triangles OKM and OLN are different, where O is the intersection point of AB and CD.

52 Prove the inequality

$$\frac{a_1+a_3}{a_1+a_2}+\frac{a_2+a_4}{a_2+a_3}+\frac{a_3+a_1}{a_3+a_4}+\frac{a_4+a_2}{a_4+a_1}\geq 4,$$

where $a_i > 0, i = 1, 2, 3, 4.$

53 Denote by $x_n(p)$ the multiplicity of the prime p in the canonical representation of the number $n!$ as a product of primes. Prove that $\frac{x_n(p)}{n} < \frac{1}{p-1}$ and $\lim_{n\to\infty}\frac{x_n(p)}{n} = \frac{1}{p-1}$.

54 A set M is formed of $\binom{2n}{n}$ men, $n = 1, 2, \ldots$. Prove that we can choose a subset P of the set M consisting of $n+1$ men such that one of the following conditions is satisfied: (1) every member of the set P knows every other member of the set P; (2) no member of the set P knows any other member of the set P.

55 Prove that the polynomial $x^4 + \lambda x^3 + \mu x^2 + \nu x + 1$ has no real roots if λ, μ, ν are real numbers satisfying

$$|\lambda| + |\mu| + |\nu| \leq \sqrt{2}$$