IMO Longlists 1972

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1 Find all integer solutions of the equation

$$1 + x + x^2 + x^3 + x^4 = y^4.$$

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2 Find all real values of the parameter a for which the system of equations

$$x^4 = yz - x^2 + a,$$

$$y^4 = zx - y^2 + a,$$

$$z^4 = xy - z^2 + a,$$

has at most one real solution.

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3 On a line a set of segments is given of total length less than n. Prove that every set of n points of the line can be translated in some direction along the line for a distance smaller than $\frac{n}{2}$ so that none of the points remain on the segments.

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4 You have a triangle, ABC. Draw in the internal angle trisectors. Let the two trisectors closest to AB intersect at D, the two trisectors closest to BC intersect at E, and the two closest to AC at F. Prove that DEF is equilateral.

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5 Given a pyramid whose base is an n-gon inscribable in a circle, let H be the projection of the top vertex of the pyramid to its base. Prove that the projections of H to the lateral edges of the pyramid lie on a circle.

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6 Prove the inequality

$$(n+1)\cos\frac{\pi}{n+1} - n\cos\frac{\pi}{n} > 1$$

for all natural numbers $n \geq 2$.

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7 f and g are real-valued functions defined on the real line. For all x and y, $f(x+y)+f(x-y) = 2f(x)g(y)$. f is not identically zero and $|f(x)| \leq 1$ for all x. Prove that $|g(x)| \leq 1$ for all x.

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8 We are given $3n$ points $A_1, A_2, \ldots, A_{3n}$ in the plane, no three of them collinear. Prove that one can construct n disjoint triangles with vertices at the points A_i.

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9 Given natural numbers k and $n, k \leq n, n \geq 3,$ find the set of all values in the interval $(0, \pi)$ that the $k^{th}-$largest among the interior angles of a convex n-gon can take.

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10 Given five points in the plane, no three of which are collinear, prove that there can be found at least two obtuse-angled triangles with vertices at the given points. Construct an example in which there are exactly two such triangles.

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11 The least number is m and the greatest number is M among $a_1, a_2, \dots, a_n$ satisfying $a_1 + a_2 + ... + a_n = 0$. Prove that

$$a_1^2 + \cdots + a_n^2 \leq -nmM$$

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12 A circle $k = (S, r)$ is given and a hexagon $AA'BB'CC'$ inscribed in it. The lengths of sides of the hexagon satisfy $AA' = A'B, BB' = B'C, CC' = C'A$. Prove that the area P of triangle ABC is not greater than the area P' of triangle $A'B'C'$. When does $P = P'$ hold?

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13 Given a sphere K, determine the set of all points A that are vertices of some parallelograms $ABCD$ that satisfy $AC \leq BD$ and whose entire diagonal BD is contained in K.

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14 (a) A plane π passes through the vertex O of the regular tetrahedron $OPQR$. We define p, q, r to be the signed distances of P, Q, R from π measured along a directed normal to π. Prove that

$$p^2 + q^2 + r^2 + (q - r)^2 + (r - p)^2 + (p - q)^2 = 2a^2,$$

where a is the length of an edge of a tetrahedron. (b) Given four parallel planes not all of which are coincident, show that a regular tetrahedron exists with a vertex on each plane.

Note: Part (b) is IMO 1972 Problem 6 (`http://www.artofproblemsolving.com/Forum/viewtopic.php?f=49\&t=60825\&start=0`)

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15 Prove that $(2m)!(2n)!$ is a multiple of $m!n!(m + n)!$ for any non-negative integers m and n.

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16 Consider the set S of all the different odd positive integers that are not multiples of 5 and that are less than $30m, m$ being a positive integer. What is the smallest integer k such that in any subset of k integers from S there must be two integers one of which divides the other? Prove your result.

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17 A solid right circular cylinder with height h and base-radius r has a solid hemisphere of radius r resting upon it. The center of the hemisphere O is on the axis of the cylinder. Let P be any point on the surface of the hemisphere and Q the point on the base circle of the cylinder that is furthest from P (measuring along the surface of the combined solid). A string is stretched over the surface from P to Q so as to be as short as possible. Show that if the string is not in a plane, the straight line PO when produced cuts the curved surface of the cylinder.

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18 We have p players participating in a tournament, each player playing against every other player exactly once. A point is scored for each victory, and there are no draws. A sequence of non-negative integers $s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_p$ is given. Show that it is possible for this sequence to be a set of final scores of the players in the tournament if and only if

$$(i) \sum_{i=1}^{p} s_i = \frac{1}{2}p(p-1)$$

and

$$(ii) \text{ for all } k < p, \sum_{i=1}^{k} s_i \geq \frac{1}{2}k(k-1).$$

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19 Let S be a subset of the real numbers with the following properties: (i) If $x \in S$ and $y \in S$, then $x - y \in S$; (ii) If $x \in S$ and $y \in S$, then $xy \in S$; (iii) S contains an exceptional number x' such that there is no number y in S satisfying $x'y + x' + y = 0$; (iv) If $x \in S$ and $x \neq x'$, there is a number y in S such that $xy + x + y = 0$.
Show that (a) S has more than one number in it; (b) $x' \neq -1$ leads to a contradiction; (c) $x \in S$ and $x \neq 0$ implies $1/x \in S$.

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20 Let n_1, n_2 be positive integers. Consider in a plane E two disjoint sets of points M_1 and M_2 consisting of $2n_1$ and $2n_2$ points, respectively, and such that no three points of the union $M_1 \cup M_2$ are collinear. Prove that there exists a straightline g with the following property: Each of the two half-planes determined by g on E (g not being included in either) contains exactly half of the points of M_1 and exactly half of the points of M_2.

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21 Prove the following assertion: The four altitudes of a tetrahedron $ABCD$ intersect in a point if and only if

$$AB^2 + CD^2 = BC^2 + AD^2 = CA^2 + BD^2.$$

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22 Show that for any $n \not\equiv 0 \pmod{10}$ there exists a multiple of n not containing the digit 0 in its decimal expansion.

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23 Does there exist a $2n$-digit number $\overline{a_{2n}a_{2n-1}\cdots a_1}$(for an arbitrary n) for which the following equality holds:

$$\overline{a_{2n}\cdots a_1} = (\overline{a_n\cdots a_1})^2?$$

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24 The diagonals of a convex 18-gon are colored in 5 different colors, each color appearing on an equal number of diagonals. The diagonals of one color are numbered $1, 2, \cdots$. One randomly chooses one-fifth of all the diagonals. Find the number of possibilities for which among the chosen diagonals there exist exactly n pairs of diagonals of the same color and with fixed indices i, j.

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25 We consider n real variables $x_i (1 \le i \le n)$, where n is an integer and $n \ge 2$. The product of these variables will be denoted by p, their sum by s, and the sum of their squares by S. Furthermore, let α be a positive constant. We now study the inequality $ps \le S\alpha$. Prove that it holds for every n-tuple (x_i) if and only if $\alpha = \frac{n+1}{2}$

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26 Find all positive real solutions to:

$$\begin{aligned}
(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) &\le 0 \\
(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) &\le 0 \\
(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) &\le 0 \\
(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) &\le 0 \\
(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) &\le 0
\end{aligned}$$

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27 Given $n > 4$, prove that every cyclic quadrilateral can be dissected into n cyclic quadrilaterals.

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28 The lengths of the sides of a rectangle are given to be odd integers. Prove that there does not exist a point within that rectangle that has integer distances to each of its four vertices.

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29 Let A, B, C be points on the sides B_1C_1, C_1A_1, A_1B_1 of a triangle $A_1B_1C_1$ such that A_1A, B_1B, C_1C are the bisectors of angles of the triangle. We have that $AC = BC$ and $A_1C_1 \neq B_1C_1$. (a) Prove that C_1 lies on the circumcircle of the triangle ABC. (b) Suppose that $\angle BAC_1 = \frac{\pi}{6}$; find the form of triangle ABC.

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30 Consider a sequence of circles $K_1, K_2, K_3, K_4, \dots$ of radii $r_1, r_2, r_3, r_4, \dots$, respectively, situated inside a triangle ABC. The circle K_1 is tangent to AB and AC; K_2 is tangent to K_1, BA, and BC; K_3 is tangent to K_2, CA, and CB; K_4 is tangent to K_3, AB, and AC; etc.
(a) Prove the relation

$$r_1 \cot \frac{1}{2}A + 2\sqrt{r_1r_2} + r_2 \cot \frac{1}{2}B = r\left(\cot \frac{1}{2}A + \cot \frac{1}{2}B\right)$$

where r is the radius of the incircle of the triangle ABC. Deduce the existence of a t_1 such that

$$r_1 = r \cot \frac{1}{2}B \cot \frac{1}{2}C \sin^2 t_1$$

(b) Prove that the sequence of circles $K_1, K_2, \dots$ is periodic.

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31 Find values of $n \in \mathbb{N}$ for which the fraction $\frac{3^n-2}{2^n-3}$ is reducible.

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32 If $n_1, n_2, \cdots, n_k$ are natural numbers and $n_1 + n_2 + \cdots + n_k = n$,
show that

$$max(n_1 n_2 \cdots n_k) = (t+1)^r t^{k-r},$$

where $t = \left[\frac{n}{k}\right]$ and r is the remainder of n upon division by k; i.e., $n = tk + r, 0 \le r \le k-1$.

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33 A rectangle $ABCD$ is given whose sides have lengths 3 and $2n$, where n is a natural number. Denote by $U(n)$ the number of ways in which one can cut the rectangle into rectangles of side lengths 1 and 2. (a) Prove that

$$U(n+1) + U(n-1) = 4U(n);$$

(b) Prove that

$$U(n) = \frac{1}{2\sqrt{3}}[(\sqrt{3}+1)(2+\sqrt{3})^n + (\sqrt{3}-1)(2-\sqrt{3})^n].$$

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34 If p is a prime number greater than 2 and a, b, c integers not divisible by p, prove that the equation

$$ax^2 + by^2 = pz + c$$

has an integer solution.

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35 (a) Prove that for $a, b, c, d \in \mathbb{R}, m \in [1, +\infty)$ with $am + b = -cm + d = m$,

$$(i)\sqrt{a^2+b^2} + \sqrt{c^2+d^2} + \sqrt{(a-c)^2+(b-d)^2} \ge \frac{4m^2}{1+m^2}, \text{ and}$$

$$(ii)2 \le \frac{4m^2}{1+m^2} < 4.$$

(b) Express a, b, c, d as functions of m so that there is equality in (i).

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36 A finite number of parallel segments in the plane are given with the property that for any three of the segments there is a line intersecting each of them. Prove that there exists a line that intersects all the given segments.

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37 On a chessboard (8×8 squares with sides of length 1) two diagonally opposite corner squares are taken away. Can the board now be covered with nonoverlapping rectangles with sides of lengths 1 and 2?

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38 Congruent rectangles with sides $m(cm)$ and $n(cm)$ are given (m, n positive integers). Characterize the rectangles that can be constructed from these rectangles (in the fashion of a jigsaw puzzle). (The number of rectangles is unbounded.)

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39 How many tangents to the curve $y = x^3 - 3x$ $(y = x^3 + px)$ can be drawn from different points in the plane?

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40 Prove the inequalities

$$\frac{u}{v} \le \frac{\sin u}{\sin v} \le \frac{\pi}{2} \times \frac{u}{v}, \text{ for } 0 \le u < v \le \frac{\pi}{2}$$

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41 The ternary expansion $x = 0.10101010\cdots$ is given. Give the binary expansion of x.
Alternatively, transform the binary expansion $y = 0.110110110\cdots$ into a ternary expansion.

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42 The decimal number 13^{101} is given. It is instead written as a ternary number. What are the two last digits of this ternary number?

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43 A fixed point A inside a circle is given. Consider all chords XY of the circle such that $\angle XAY$ is a right angle, and for all such chords construct the point M symmetric to A with respect to XY . Find the locus of points M.

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44 Prove that from a set of ten distinct two-digit numbers, it is always possible to find two disjoint subsets whose members have the same sum.

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45 Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD intersect at point O. Let a line through O intersect segment AB at M and segment CD at N. Prove that the segment MN is not longer than at least one of the segments AC and BD.

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46 Numbers $1, 2, \cdots, 16$ are written in a 4×4 square matrix so that the sum of the numbers in every row, every column, and every diagonal is the same and furthermore that the numbers 1 and 16 lie in opposite corners. Prove that the sum of any two numbers symmetric with respect to the center of the square equals 17.
