IMO Longlists 1973

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1 Find the maximal positive number r with the following property: If all altitudes of a tetrahedron are ≥ 1, then a sphere of radius r fits into the tetrahedron.

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2 Let OX, OY and OZ be three rays in the space, and G a point "*between these rays*" (i. e. in the interior of the part of the space bordered by the angles YOZ, ZOX and XOY). Consider a plane passing through G and meeting the rays OX, OY and OZ in the points A, B, C, respectively. There are infinitely many such planes; construct the one which minimizes the volume of the tetrahedron $OABC$.

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3 Is the number

$$\sqrt[3]{\sqrt{5}+2}+\sqrt[3]{\sqrt{5}-2}$$

rational or irrational?

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4 A circle of radius 1 is placed in a corner of a room (i.e., it touches the horizontal floor and two vertical walls perpendicular to each other). Find the locus of the center of the band for all of its possible positions.

Note. For the solution of this problem, it is useful to know the following Monge theorem: The locus of all points P, such that the two tangents from P to the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are perpendicular to each other, is a circle a so-called Monge circle with equation $x^2+y^2=a^2+b^2$.

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5 Given a ball K. Find the locus of the vertices A of all parallelograms $ABCD$ such that $AC \leq BD$, and the diagonal BD lies completely inside the ball K.

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6 Let $P_i(x_i, y_i)$ (with $i = 1, 2, 3, 4, 5$) be five points with integer coordinates, no three collinear. Show that among all triangles with vertices at these points, at least three have integer areas.

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7 Given a tetrahedron $ABCD$. Let $x = AB \cdot CD, y = AC \cdot BD$ and $z = AD \cdot BC$. Prove that there exists a triangle with the side lengths x, y and z.

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8 Let a be a non-zero real number. For each integer n, we define $S_n = a^n + a^{-n}$. Prove that if for some integer k, the sums S_k and S_{k+1} are integers, then the sums S_n are integers for all integers n.

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9 Prove that $2^{147} - 1$ is divisible by 343.

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