IMO Longlists 1976

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1 Let ABC be a triangle with bisectors AA_1, BB_1, CC_1 ($A_1 \in BC$, etc.) and M their common point. Consider the triangles $MB_1A, MC_1A, MC_1B, MA_1B, MA_1C, MB_1C$, and their inscribed circles. Prove that if four of these six inscribed circles have equal radii, then $AB = BC = CA$.

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2 Let P be a set of n points and S a set of l segments. It is known that: (i) No four points of P are coplanar. (ii) Any segment from S has its endpoints at P. (iii) There is a point, say g, in P that is the endpoint of a maximal number of segments from S and that is not a vertex of a tetrahedron having all its edges in S.
Prove that $l \leq \frac{n^2}{3}$

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3 Let $a_0, a_1, \ldots, a_n, a_{n+1}$ be a sequence of real numbers satisfying the following conditions:

$$a_0 = a_{n+1} = 0,$$
$$|a_{k-1} - 2a_k + a_{k+1}| \leq 1 \quad (k = 1, 2, \ldots, n).$$

Prove that $|a_k| \leq \frac{k(n+1-k)}{2} \quad (k = 0, 1, \ldots, n+1)$.

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4 Find all pairs of natural numbers (m, n) for which $2^m 3^n + 1$ is the square of some integer.

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5 Let $ABCDS$ be a pyramid with four faces and with $ABCD$ as a base, and let a plane α through the vertex A meet its edges SB and SD at points M and N, respectively. Prove that if the intersection of the plane α with the pyramid $ABCDS$ is a parallelogram, then $SM \cdot SN > BM \cdot DN$.

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6 For each point X of a given polytope, denote by $f(X)$ the sum of the distances of the point X from all the planes of the faces of the polytope. Prove that if f attains its maximum at an interior point of the polytope, then f is constant.

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7 Let P be a fixed point and T a given triangle that contains the point P. Translate the triangle T by a given vector $\mathbf{v}$ and denote by T' this new triangle. Let r, R, respectively, be the radii of the smallest disks centered at P that contain the triangles T, T', respectively. Prove that $r + |\mathbf{v}| \leq 3R$ and find an example to show that equality can occur.

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8 In a convex quadrilateral (in the plane) with the area of 32 cm^2 the sum of two opposite sides and a diagonal is 16 cm. Determine all the possible values that the other diagonal can have.

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9 Find all (real) solutions of the system

$$3x_1 - x_2 - x_3 - x_5 = 0,$$
$$-x_1 + 3x_2 - x_4 - x_6 = 0,$$
$$-x_1 + 3x_3 - x_4 - x_7 = 0,$$
$$-x_2 - x_3 + 3x_4 - x_8 = 0,$$
$$-x_1 + 3x_5 - x_6 - x_7 = 0,$$
$$-x_2 - x_5 + 3x_6 - x_8 = 0,$$
$$-x_3 - x_5 + 3x_7 - x_8 = 0,$$
$$-x_4 - x_6 - x_7 + 3x_8 = 0.$$

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10 Show that the reciprocal of any number of the form $2(m^2+m+1)$, where m is a positive integer, can be represented as a sum of consecutive terms in the sequence $(a_j)_{j=1}^{\infty}$

$$a_j = \frac{1}{j(j+1)(j+2)}$$

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11 Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for j$= 2, \dots$ Prove that for any positive integer n the roots of the equation $P_n(x) = x$ are all real and distinct.

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12 Five points lie on the surface of a ball of unit radius. Find the maximum of the smallest distance between any two of them.

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13 A sequence (u_n) is defined by

$$u_0 = 2 \quad u_1 = \frac{5}{2}, u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1 \quad \text{for } n = 1, \dots$$

Prove that for any positive integer n we have

$$[u_n] = 2^{\frac{(2^n - (-1)^n)}{3}}$$

(where $[x]$ denotes the smallest integer $\leq x$)

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14 A sequence $\{u_n\}$ of integers is defined by

$$u_1 = 2, u_2 = u_3 = 7,$$

$$u_{n+1} = u_n u_{n-1} - u_{n-2}, \text{ for } n \geq 3$$

Prove that for each $n \geq 1$, u_n differs by 2 from an integral square.

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15 Let ABC and $A'B'C'$ be any two coplanar triangles. Let L be a point such that $AL||BC, A'L||B'C'$, and M, N similarly defined. The line BC meets $B'C'$ at P, and similarly defined are Q and R. Prove that PL, QM, RN are concurrent.

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16 Prove that there is a positive integer n such that the decimal representation of 7^n contains a block of at least m consecutive zeros, where m is any given positive integer.

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17 Show that there exists a convex polyhedron with all its vertices on the surface of a sphere and with all its faces congruent isosceles triangles whose ratio of sides are $\sqrt{3} : \sqrt{3} : 2$.

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18 Prove that the number $19^{1976} + 76^{1976}$: (a) is divisible by the (Fermat) prime number $F_4 = 2^{2^4} + 1$; (b) is divisible by at least four distinct primes other than F_4.

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19 For a positive integer n, let $6^{(n)}$ be the natural number whose decimal representation consists of n digits 6. Let us define, for all natural numbers m, k with $1 \le k \le m$

$$\left[\begin{array}{c} m \\ k \end{array}\right] = \frac{6^{(m)}6^{(m-1)}\cdots 6^{(m-k+1)}}{6^{(1)}6^{(2)}\cdots 6^{(k)}}.$$

Prove that for all m, k, $\left[\begin{array}{c} m \\ k \end{array}\right]$ is a natural number whose decimal representation consists of exactly $k(m + k - 1) - 1$ digits.

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20 Let $(a_n), n = 0, 1, ...,$ be a sequence of real numbers such that $a_0 = 0$ and

$$a_{n+1}^3 = \frac{1}{2}a_n^2 - 1, n = 0, 1, \cdots$$

Prove that there exists a positive number $q, q < 1$, such that for all $n = 1, 2, \dots,$

$$|a_{n+1} - a_n| \le q|a_n - a_{n-1}|,$$

and give one such q explicitly.

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21 Find the largest positive real number p (if it exists) such that the inequality

$$x_1^2 + x_2^2 + \cdots + x_n^2 \ge p(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n)$$

is satisfied for all real numbers x_i, and $(a) n = 2$; $(b) n = 5$.
Find the largest positive real number p (if it exists) such that the inequality holds for all real numbers x_i and all natural numbers $n, n \ge 2$.

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22 A regular pentagon $A_1A_2A_3A_4A_5$ with side length s is given. At each point A_i, a sphere K_i of radius $\frac{s}{2}$ is constructed. There are two spheres K_1 and K_2 each of radius $\frac{s}{2}$ touching all the five spheres K_i. Decide whether K_1 and K_2 intersect each other, touch each other, or have no common points.

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23 Prove that in a Euclidean plane there are infinitely many concentric circles C such that all triangles inscribed in C have at least one irrational side.

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24 Let $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$. Prove that for all $A \geq 1$, there exists an interval I of length $2\sqrt[n]{A}$ such that for all $x \in I$,

$$|(x-x_1)(x-x_2)\cdots(x-x_n)| \leq A.$$

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25 We consider the following system
with $q = 2p$:

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1q}x_q &= 0, \\ a_{21}x_1 + \ldots + a_{2q}x_q &= 0, \\ &\ldots, \\ a_{p1}x_1 + \ldots + a_{pq}x_q &= 0, \end{aligned}$$

in which every coefficient is an element from the set $\{-1, 0, 1\}$. Prove that there exists a solution $x_1, \ldots, x_q$ for the system with the properties:

a.) all $x_j, j = 1, \ldots, q$ are integers;

b.) there exists at least one j for which $x_j \neq 0$;

c.) $|x_j| \leq q$ for any $j = 1, \ldots, q$.

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26 A box whose shape is a parallelepiped can be completely filled with cubes of side 1. If we put in it the maximum possible number of cubes, each of volume 2, with the sides parallel to those of the box, then exactly 40 percent of the volume of the box is occupied. Determine the possible dimensions of the box.

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27 In a plane three points P, Q, R, not on a line, are given. Let k, l, m be positive numbers. Construct a triangle ABC whose sides pass through P, Q, and R such that P divides the segment AB in the ratio $1 : k$, Q divides the segment BC in the ratio $1 : l$, and R divides the segment CA in the ratio $1 : m$.

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28 Let Q be a unit square in the plane: $Q = [0,1] \times [0,1]$. Let $T : Q \longrightarrow Q$ be defined as follows:

$$T(x,y) = \begin{cases} (2x, \frac{y}{2}) & \text{if } 0 \leq x \leq \frac{1}{2}; \\ (2x-1, \frac{y}{2} + \frac{1}{2}) & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Show that for every disk $D \subset Q$ there exists an integer $n > 0$ such that $T^n(D) \cap D \neq \emptyset$.

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29 Let $I = (0, 1]$ be the unit interval of the real line. For a given number $a \in (0, 1)$ we define a map $T : I \to I$ by the formula
if

$$T(x, y) = \begin{cases} x + (1 - a), & \text{if } 0 < x \leq a, \\ \\ x - a, & \text{if } a < x \leq 1. \end{cases}$$

Show that for every interval $J \subset I$ there exists an integer $n > 0$ such that $T^n(J) \cap J \neq \emptyset$.

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30 Prove that if $P(x) = (x - a)^k Q(x)$, where k is a positive integer, a is a nonzero real number, $Q(x)$ is a nonzero polynomial, then $P(x)$ has at least $k + 1$ nonzero coefficients.

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31 Into every lateral face of a quadrangular pyramid a circle is inscribed. The circles inscribed into adjacent faces are tangent (have one point in common). Prove that the points of contact of the circles with the base of the pyramid lie on a circle.

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32 We consider the infinite chessboard covering the whole plane. In every field of the chessboard there is a nonnegative real number. Every number is the arithmetic mean of the numbers in the four adjacent fields of the chessboard. Prove that the numbers occurring in the fields of the chessboard are all equal.

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33 A finite set of points P in the plane has the following property: Every line through two points in P contains at least one more point belonging to P. Prove that all points in P lie on a straight line.

This may be a well known theorem called "Sylvester Gallai", but I didn't find this problem (I mean, exactly this one) using search function. So please discuss about the problem here, in this topic. Thanks :)

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34 Let $\{a_n\}_0^\infty$ and $\{b_n\}_0^\infty$ be two sequences determined by the recursion formulas

$$a_{n+1} = a_n + b_n,$$

$$b_{n+1} = 3a_n + b_n, n = 0, 1, 2, \cdots,$$

and the initial values $a_0 = b_0 = 1$. Prove that there exists a uniquely determined constant c such that $n|ca_n - b_n| < 2$ for all nonnegative integers n.

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35 Let P be a polynomial with real coefficients such that $P(x) > 0$ if $x > 0$. Prove that there exist polynomials Q and R with nonnegative coefficients such that $P(x) = \frac{Q(x)}{R(x)}$ if $x > 0$.

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36 Three concentric circles with common center O are cut by a common chord in successive points A, B, C. Tangents drawn to the circles at the points A, B, C enclose a triangular region. If the distance from point O to the common chord is equal to p, prove that the area of the region enclosed by the tangents is equal to

$$\frac{AB \cdot BC \cdot CA}{2p}$$

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37 From a square board 11 squares long and 11 squares wide, the central square is removed. Prove that the remaining 120 squares cannot be covered by 15 strips each 8 units long and one unit wide.

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38 Let $x = \sqrt{a} + \sqrt{b}$, where a and b are natural numbers, x is not an integer, and $x < 1976$. Prove that the fractional part of x exceeds $10^{-19.76}$.

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39 In ABC, the inscribed circle is tangent to side BC at$X$. Segment AX is drawn. Prove that the line joining the midpoint of AX to the midpoint of side BC passes through center I of the inscribed circle.

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40 Let $g(x)$ be a fixed polynomial with real coefficients and define $f(x)$ by $f(x) = x^2 + xg(x^3)$. Show that $f(x)$ is not divisible by $x^2 - x + 1$.

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41 Determine the greatest number, who is the product of some positive integers, and the sum of these numbers is 1976.

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42 For a point O inside a triangle ABC, denote by $A_1, B_1, C_1,$ the respective intersection points of AO, BO, CO with the corresponding sides. Let

$$n_1 = \frac{AO}{A_1O}, n_2 = \frac{BO}{B_1O}, n_3 = \frac{CO}{C_1O}.$$

What possible values of n_1, n_2, n_3 can all be positive integers?

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43 Prove that if for a polynomial $P(x, y)$, we have

$$P(x - 1, y - 2x + 1) = P(x, y),$$

then there exists a polynomial $\Phi(x)$ with $P(x, y) = \Phi(y - x^2)$.

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44 A circle of radius 1 rolls around a circle of radius $\sqrt{2}$. Initially, the tangent point is colored red. Afterwards, the red points map from one circle to another by contact. How many red points will be on the bigger circle when the center of the smaller one has made n circuits around the bigger one?

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45 We are given $n (n \geq 5)$ circles in a plane. Suppose that every three of them have a common point. Prove that all n circles have a common point.

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46 Let a, b, c, d be nonnegative real numbers. Prove that

$$a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2.$$

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47 Prove that 5^n has a block of 1976 consecutive $0's$ in its decimal representation.

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48 The polynomial $1976(x + x^2 + \cdots + x^n)$ is decomposed into a sum of polynomials of the form $a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_1, a_2, \ldots, a_n$ are distinct positive integers not greater than n. Find all values of n for which such a decomposition is possible.

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49 Determine whether there exist 1976 nonsimilar triangles with angles α, β, γ, each of them satisfying the relations

$$\frac{\sin\alpha + \sin\beta + \sin\gamma}{\cos\alpha + \cos\beta + \cos\gamma} = \frac{12}{7} \text{ and } \sin\alpha\sin\beta\sin\gamma = \frac{12}{25}$$

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50 Find a function $f(x)$ defined for all real values of x such that for all x,

$$f(x + 2) - f(x) = x^2 + 2x + 4,$$

and if $x \in [0, 2)$, then $f(x) = x^2$.

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51 Four swallows are catching a fly. At first, the swallows are at the four vertices of a tetrahedron, and the fly is in its interior. Their maximal speeds are equal. Prove that the swallows can catch the fly.
