IMO Longlists 1977

www.artofproblemsolving.com/community/c4009

by WakeUp, dondigo, Amir Hossein, orl, enescu

1 A pentagon $ABCDE$ inscribed in a circle for which $BC < CD$ and $AB < DE$ is the base of a pyramid with vertex S. If AS is the longest edge starting from S, prove that $BS > CS$.

---

2 Find all functions $f:\mathbb{N}\to\mathbb{N}$ satisfying following condition:

$$f(n+1) > f(f(n)), \quad \forall n \in \mathbb{N}.$$

---

3 In a company of n persons, each person has no more than d acquaintances, and in that company there exists a group of k persons, $k \geq d$, who are not acquainted with each other. Prove that the number of acquainted pairs is not greater than $\left[\frac{n^2}{4}\right]$.

---

4 We are given n points in space. Some pairs of these points are connected by line segments so that the number of segments equals $[n^2/4]$, and a connected triangle exists. Prove that any point from which the maximal number of segments starts is a vertex of a connected triangle.

---

5 A lattice point in the plane is a point both of whose coordinates are integers. Each lattice point has four neighboring points: upper, lower, left, and right. Let k be a circle with radius $r \geq 2$, that does not pass through any lattice point. An interior boundary point is a lattice point lying inside the circle k that has a neighboring point lying outside k. Similarly, an exterior boundary point is a lattice point lying outside the circle k that has a neighboring point lying inside k. Prove that there are four more exterior boundary points than interior boundary points.

---

6 Let $x_1, x_2, \ldots, x_n$ $(n \geq 1)$ be real numbers such that $0 \leq x_j \leq \pi,\ j = 1, 2, \ldots, n$. Prove that if $\sum_{j=1}^n (\cos x_j + 1)$ is an odd integer, then $\sum_{j=1}^n \sin x_j \geq 1$.

---

7 Prove the following assertion: If $c_1, c_2, \ldots, c_n$ $(n \geq 2)$ are real numbers such that

$$(n-1)(c_1^2 + c_2^2 + \cdots + c_n^2) = (c_1 + c_2 + \cdots + c_n)^2,$$

then either all these numbers are nonnegative or all these numbers are nonpositive.

---

8 A hexahedron $ABCDE$ is made of two regular congruent tetrahedra $ABCD$ and $ABCE$. Prove that there exists only one isometry $\mathbf{Z}$ that maps points A, B, C, D, E onto B, C, A, E, D, respectively. Find all points X on the surface of hexahedron whose distance from $\mathbf{Z}(X)$ is minimal.

---

9 Let $ABCD$ be a regular tetrahedron and $\mathbf{Z}$ an isometry mapping A, B, C, D into B, C, D, A, respectively. Find the set M of all points X of the face ABC whose distance from $\mathbf{Z}(X)$ is equal to a given number t. Find necessary and sufficient conditions for the set M to be nonempty.

---

10 Let a, b be two natural numbers. When we divide $a^2 + b^2$ by $a + b$, we the the remainder r and the quotient q. Determine all pairs (a, b) for which $q^2 + r = 1977$.

---

11 Let n and z be integers greater than 1 and $(n, z) = 1$. Prove:
(a) At least one of the numbers $z_i = 1 + z + z^2 + \cdots + z^i,\ i = 0, 1, \ldots, n-1$, is divisible by n.
(b) If $(z-1, n) = 1$, then at least one of the numbers z_i is divisible by n.

---

12 Let z be an integer > 1 and let M be the set of all numbers of the form $z_k = 1+z+\cdots+z^k,\ k = 0, 1, \ldots$. Determine the set T of divisors of at least one of the numbers z_k from M.

---

13 Describe all closed bounded figures Φ in the plane any two points of which are connectable by a semicircle lying in Φ.

---

14 There are 2^n words of length n over the alphabet $\{0, 1\}$. Prove that the following algorithm generates the sequence $w_0, w_1, \ldots, w_{2^n-1}$ of all these words such that any two consecutive words differ in exactly one digit.

(1) $w_0 = 00\ldots0$ (n zeros).

(2) Suppose $w_{m-1} = a_1a_2\ldots a_n, \quad a_i \in \{0, 1\}$. Let $e(m)$ be the exponent of 2 in the representation of n as a product of primes, and let $j = 1 + e(m)$. Replace the digit a_j in the word w_{m-1} by $1 - a_j$. The obtained word is w_m.

---

15 Let n be an integer greater than 1. In the Cartesian coordinate system we consider all squares with integer vertices (x, y) such that $1 \le x, y \le n$. Denote by p_k $(k = 0, 1, 2, \ldots)$ the number of pairs of points that are vertices of exactly k such squares. Prove that $\sum_k (k-1)p_k = 0$.

---

16 Let n be a positive integer. How many integer solutions $(i, j, k, l),\ 1 \le i, j, k, l \le n$, does the following system of inequalities have:

$$1 \le -j + k + l \le n$$
$$1 \le i - k + l \le n$$
$$1 \le i - j + l \le n$$
$$1 \le i + j - k \le n\,?$$

---

17 A ball K of radius r is touched from the outside by mutually equal balls of radius R. Two of these balls are tangent to each other. Moreover, for two balls K_1 and K_2 tangent to K and tangent to each other there exist two other balls tangent to K_1, K_2 and also to K. How many balls are tangent to K? For a given r determine R.

---

18 Given an isosceles triangle ABC with a right angle at C, construct the center M and radius r of a circle cutting on segments AB, BC, CA the segments DE, FG, and HK, respectively, such that $\angle DME + \angle FMG + \angle HMK = 180^\circ$ and $DE : FG : HK = AB : BC : CA$.

---

19 Given any integer $m > 1$ prove that there exist infinitely many positive integers n such that the last m digits of 5^n are a sequence $a_m, a_{m-1}, \dots, a_1 = 5$ $(0 \le a_j < 10)$ in which each digit except the last is of opposite parity to its successor (i.e., if a_i is even, then a_{i-1} is odd, and if a_i is odd, then a_{i-1} is even).

---

20 Let a, b, A, B be given reals. We consider the function defined by

$$f(x) = 1 - a \cdot \cos(x) - b \cdot \sin(x) - A \cdot \cos(2x) - B \cdot \sin(2x).$$

Prove that if for any real number x we have $f(x) \geq 0$ then $a^2 + b^2 \leq 2$ and $A^2 + B^2 \leq 1.$

---

21 Given that $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = x_1y_1 + x_2y_2 + x_3y_3 = 0,$ prove that:

$$\frac{x_1^2}{x_1^2 + x_2^2 + x_3^2} + \frac{y_1^2}{y_1^2 + y_2^2 + y_3^2} = \frac{2}{3}$$

---

22 Let S be a convex quadrilateral $ABCD$ and O a point inside it. The feet of the perpendiculars from O to AB, BC, CD, DA are A_1, B_1, C_1, D_1 respectively. The feet of the perpendiculars from O to the sides of S_i, the quadrilateral $A_iB_iC_iD_i$, are $A_{i+1}B_{i+1}C_{i+1}D_{i+1}$, where $i = 1, 2, 3.$ Prove that S_4 is similar to S.

---

23 For which positive integers n do there exist two polynomials f and g with integer coefficients of n variables $x_1, x_2, \dots, x_n$ such that the following equality is satisfied:

$$\sum_{i=1}^{n} x_i f(x_1, x_2, \dots, x_n) = g(x_1^2, x_2^2, \dots, x_n^2) \ ?$$

---

24 Determine all real functions $f(x)$ that are defined and continuous on the interval $(-1, 1)$ and that satisfy the functional equation

$$f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)} \qquad (x, y, x + y \in (-1, 1)).$$

---

25 Prove the identity

$$(z+a)^n = z^n + a\sum_{k=1}^{n}\binom{n}{k}(a-kb)^{k-1}(z+kb)^{n-k}$$

---

26 Let p be a prime number greater than 5. Let V be the collection of all positive integers n that can be written in the form $n = kp+1$ or $n = kp-1$ $(k = 1, 2, \ldots)$. A number $n \in V$ is called *indecomposable* in V if it is impossible to find $k, l \in V$ such that $n = kl$. Prove that there exists a number $N \in V$ that can be factorized into indecomposable factors in V in more than one way.

---

27 Let n be a given number greater than 2. We consider the set V_n of all the integers of the form $1 + kn$ with $k = 1, 2, \ldots$ A number m from V_n is called indecomposable in V_n if there are not two numbers p and q from V_n so that $m = pq$. Prove that there exist a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Expressions which differ only in order of the elements of V_n will be considered the same.)

---

28 Let n be an integer greater than 1. Define

$$x_1 = n, y_1 = 1, x_{i+1} = \left[\frac{x_i + y_i}{2}\right], y_{i+1} = \left[\frac{n}{x_{i+1}}\right], \qquad \text{for } i = 1, 2, \ldots ,$$

where $[z]$ denotes the largest integer less than or equal to z. Prove that

$$\min\{x_1, x_2, \ldots, x_n\} = [\sqrt{n}]$$

---

29 In the interior of a square $ABCD$ we construct the equilateral triangles ABK, BCL, CDM, DAN. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments $AK, BK, BL, CL, CM, DM, DN, AN$ are the 12 vertices of a regular dodecagon.

---

30 A triangle ABC with $\angle A = 30^\circ$ and $\angle C = 54^\circ$ is given. On BC a point D is chosen such that $\angle CAD = 12^\circ$. On AB a point E is chosen such that $\angle ACE = 6^\circ$. Let S be the point of intersection of AD and CE. Prove that $BS = BC$.

---

31 Let f be a function defined on the set of pairs of nonzero rational numbers whose values are positive real numbers. Suppose that f satisfies the following conditions:

(1) $f(ab, c) = f(a, c)f(b, c),\ f(c, ab) = f(c, a)f(c, b);$

(2) $f(a, 1-a) = 1$

Prove that $f(a, a) = f(a, -a) = 1, f(a, b)f(b, a) = 1.$

---

32 In a room there are nine men. Among every three of them there are two mutually acquainted. Prove that some four of them are mutually acquainted.

---

33 A circle K centered at $(0,0)$ is given. Prove that for every vector (a_1, a_2) there is a positive integer n such that the circle K translated by the vector $n(a_1, a_2)$ contains a lattice point (i.e., a point both of whose coordinates are integers).

---

34 Let B be a set of k sequences each having n terms equal to 1 or -1. The product of two such sequences $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ is defined as $(a_1b_1, a_2b_2, \dots, a_nb_n)$. Prove that there exists a sequence $(c_1, c_2, \dots, c_n)$ such that the intersection of B and the set containing all sequences from B multiplied by $(c_1, c_2, \dots, c_n)$ contains at most $\frac{k^2}{2^n}$ sequences.

---

35 Find all numbers $N = \overline{a_1a_2\dots a_n}$ for which $9 \times \overline{a_1a_2\dots a_n} = \overline{a_n\dots a_2a_1}$ such that at most one of the digits $a_1, a_2, \dots, a_n$ is zero.

---

36 Consider a sequence of numbers $(a_1, a_2, \dots, a_{2^n})$. Define the operation

$$S\Big((a_1, a_2, \dots, a_{2^n})\Big) = (a_1a_2, a_2a_3, \dots, a_{2^{n-1}a_{2^n}, a_{2^n}a_1}).$$

Prove that whatever the sequence $(a_1, a_2, \dots, a_{2^n})$ is, with $a_i \in \{-1, 1\}$ for $i = 1, 2, \dots, 2^n$, after finitely many applications of the operation we get the sequence $(1, 1, \dots, 1)$.

---

37 Let $A_1, A_2, \dots, A_{n+1}$ be positive integers such that $(A_i, A_{n+1}) = 1$ for every $i = 1, 2, \dots, n$. Show that the equation

$$x_1^{A_1} + x_2^{A_2} + \dots + x_n^{A_n} = x_{n+1}^{A_{n+1}}$$

has an infinite set of solutions $(x_1, x_2, \dots, x_{n+1})$ in positive integers.

---

38 Let $m_j > 0$ for $j = 1, 2, \dots, n$ and $a_1 \le \dots \le a_n < b_1 \le \dots \le b_n < c_1 \le \dots \le c_n$ be real numbers. Prove that

$$\left(\sum_{j=1}^{n} m_j(a_j + b_j + c_j)\right)^2 > 3\left(\sum_{j=1}^{n} m_j\right)\left(\sum_{j=1}^{n} m_j(a_jb_j + b_jc_j + c_ja_j)\right).$$

---

39 Consider 37 distinct points in space, all with integer coordinates. Prove that we may find among them three distinct points such that their barycentre has integers coordinates.

---

40 The numbers $1, 2, 3, \dots, 64$ are placed on a chessboard, one number in each square. Consider all squares on the chessboard of size 2×2. Prove that there are at least three such squares for which the sum of the 4 numbers contained exceeds 100.

---

41 A wheel consists of a fixed circular disk and a mobile circular ring. On the disk the numbers $1, 2, 3, \dots, N$ are marked, and on the ring N integers $a_1, a_2, \dots, a_N$ of sum 1 are marked. The ring can be turned into N different positions in which the numbers on the disk and on the ring match each other. Multiply every number on the ring with the corresponding number on the disk and form the sum of N products. In this way a sum is obtained for every position of the ring. Prove that the N sums are different.

---

42 The sequence $a_{n,k}$, $k = 1, 2, 3, \dots, 2^n$, $n = 0, 1, 2, \dots$, is defined by the following recurrence formula:

$$a_1 = 2, \qquad a_{n,k} = 2a^3_{n-1,k}, \qquad , a_{n,k+2^{n-1}} = \frac{1}{2}a^3_{n-1,k}$$

$$\text{for} \quad k = 1, 2, 3, \dots, 2^{n-1} \text{ }, n = 0, 1, 2, \dots$$

Prove that the numbers $a_{n,k}$ are all different.

---

43 Evaluate

$$S = \sum_{k=1}^{n} k(k+1)\dots(k+p),$$

where n and p are positive integers.

---

44 Let E be a finite set of points in space such that E is not contained in a plane and no three points of E are collinear. Show that E contains the vertices of a tetrahedron $T = ABCD$ such that $T \cap E = \{A, B, C, D\}$ (including interior points of T) and such that the projection of A onto the plane BCD is inside a triangle that is similar to the triangle BCD and whose sides have midpoints B, C, D.

---

45 Let E be a finite set of points such that E is not contained in a plane and no three points of E are collinear. Show that at least one of the following alternatives holds:

(i) E contains five points that are vertices of a convex pyramid having no other points in common with E;

(ii) some plane contains exactly three points from E.

---

46 Let f be a strictly increasing function defined on the set of real numbers. For x real and t positive, set

$$g(x, t) = \frac{f(x+t) - f(x)}{f(x) - f(x-t)}.$$

Assume that the inequalities

$$2^{-1} < g(x, t) < 2$$

hold for all positive t if $x = 0$, and for all $t \leq |x|$ otherwise.
Show that

$$14^{-1} < g(x, t) < 14$$

for all real x and positive t.

47 A square $ABCD$ is given. A line passing through A intersects CD at Q. Draw a line parallel to AQ that intersects the boundary of the square at points M and N such that the area of the quadrilateral $AMNQ$ is maximal.

48 The intersection of a plane with a regular tetrahedron with edge a is a quadrilateral with perimeter P. Prove that $2a \leq P \leq 3a$.

49 Find all pairs of integers (p, q) for which all roots of the trinomials $x^2 + px + q$ and $x^2 + qx + p$ are integers.

50 Determine all positive integers n for which there exists a polynomial $P_n(x)$ of degree n with integer coefficients that is equal to n at n different integer points and that equals zero at zero.

51 Several segments, which we shall call white, are given, and the sum of their lengths is 1. Several other segments, which we shall call black, are given, and the sum of their lengths is 1. Prove that every such system of segments can be distributed on the segment that is 1.51 long in the following way: Segments of the same colour are disjoint, and segments of different colours are either disjoint or one is inside the other. Prove that there exists a system that cannot be distributed in that way on the segment that is 1.49 long.

52 Two perpendicular chords are drawn through a given interior point P of a circle with radius R. Determine, with proof, the maximum and the minimum of the sum of the lengths of these two chords if the distance from P to the center of the circle is kR.

53 Find all pairs of integers a and b for which

$$7a + 14b = 5a^2 + 5ab + 5b^2$$

54 If $0 \leq a \leq b \leq c \leq d$, prove that

$$a^b b^c c^d d^a \geq b^a c^b d^c a^d.$$

55 Through a point O on the diagonal BD of a parallelogram $ABCD$, segments MN parallel to AB, and PQ parallel to AD, are drawn, with M on AD, and Q on AB. Prove that diagonals AO, BP, DN (extended if necessary) will be concurrent.

56 The four circumcircles of the four faces of a tetrahedron have equal radii. Prove that the four faces of the tetrahedron are congruent triangles.

57 In a finite sequence of real numbers the sum of any seven successive terms is negative and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

---

58 Prove that for every triangle the following inequality holds:

$$\frac{ab+bc+ca}{4S} \geq \cot\frac{\pi}{6}.$$

where a, b, c are lengths of the sides and S is the area of the triangle.

---

59 Let E be a set of n points in the plane $(n \geq 3)$ whose coordinates are integers such that any three points from E are vertices of a nondegenerate triangle whose centroid doesnt have both coordinates integers. Determine the maximal n.

---

60 Suppose $x_0, x_1, \dots, x_n$ are integers and $x_0 > x_1 > \dots > x_n$. Prove that at least one of the numbers $|F(x_0)|, |F(x_1)|, |F(x_2)|, \dots, |F(x_n)|$, where

$$F(x) = x^n + a_1x^{n-1} + \dots + a_n, \quad a_i \in \mathbb{R}, \quad i = 1, \dots, n,$$

is greater than $\frac{n!}{2^n}$.
