IMO Longlists 1978

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1 The set $M = \{1, 2, ..., 2n\}$ is partitioned into k nonintersecting subsets $M_1, M_2, \dots, M_k$, where $n \ge k^3 + k$. Prove that there exist even numbers $2j_1, 2j_2, \dots, 2j_{k+1}$ in M that are in one and the same subset M_i $(1 \le i \le k)$ such that the numbers $2j_1 - 1, 2j_2 - 1, \dots, 2j_{k+1} - 1$ are also in one and the same subset $M_j (1 \le j \le k)$.

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2 If

$$f(x) = (x + 2x^2 + \cdots + nx^n)^2 = a_2x^2 + a_3x^3 + \cdots + a_{2n}x^{2n},$$

prove that

$$a_{n+1} + a_{n+2} + \cdots + a_{2n} = \binom{n+1}{2} \frac{5n^2 + 5n + 2}{12}$$

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3 Find all numbers α for which the equation

$$x^2 - 2x[x] + x - \alpha = 0$$

has two nonnegative roots. ($[x]$ denotes the largest integer less than or equal to x.)

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4 Two identically oriented equilateral triangles, ABC with center S and $A'B'C$, are given in the plane. We also have $A' \neq S$ and $B' \neq S$. If M is the midpoint of $A'B$ and N the midpoint of AB', prove that the triangles $SB'M$ and $SA'N$ are similar.

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5 Prove that for any triangle ABC there exists a point P in the plane of the triangle and three points A', B' , and C' on the lines BC, AC, and AB respectively such that

$$AB \cdot PC' = AC \cdot PB' = BC \cdot PA' = 0.3M^2,$$

where $M = max\{AB, AC, BC\}$.

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6 Prove that for all $X > 1$, there exists a triangle whose sides have lengths $P_1(X) = X^4 + X^3 + 2X^2 + X + 1, P_2(X) = 2X^3 + X^2 + 2X + 1$, and $P_3(X) = X^4 - 1$. Prove that all these triangles have the same greatest angle and calculate it.

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7 Let m and n be positive integers such that $1 \le m < n$. In their decimal representations, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n. Find m and n such that $m + n$ has its least value.

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8 For two given triangles $A_1A_2A_3$ and $B_1B_2B_3$ with areas Δ_A and Δ_B, respectively, $A_iA_k \geq B_iB_k, i, k = 1, 2, 3$. Prove that $\Delta_A \geq \Delta_B$ if the triangle $A_1A_2A_3$ is not obtuse-angled.

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9 Let T_1 be a triangle having a, b, c as lengths of its sides and let T_2 be another triangle having u, v, w as lengths of its sides. If P, Q are the areas of the two triangles, prove that

$$16PQ \leq a^2(-u^2 + v^2 + w^2) + b^2(u^2 - v^2 + w^2) + c^2(u^2 + v^2 - w^2).$$

When does equality hold?

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10 Show that for any natural number n there exist two prime numbers p and $q, p \neq q$, such that n divides their difference.

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11 Find all natural numbers $n < 1978$ with the following property: If m is a natural number, $1 < m < n$, and $(m, n) = 1$ (i.e., m and n are relatively prime), then m is a prime number.

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12 The equation $x^3 + ax^2 + bx + c = 0$ has three (not necessarily distinct) real roots t, u, v. For which a, b, c do the numbers t^3, u^3, v^3 satisfy the equation $x^3 + a^3x^2 + b^3x + c^3 = 0$?

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13 The satellites A and B circle the Earth in the equatorial plane at altitude h. They are separated by distance $2r$, where r is the radius of the Earth. For which h can they be seen in mutually perpendicular directions from some point on the equator?

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14 Let $p(x, y)$ and $q(x, y)$ be polynomials in two variables such that for $x \geq 0, y \geq 0$ the following conditions hold: $(i) p(x, y)$ and $q(x, y)$ are increasing functions of x for every fixed y. $(ii) p(x, y)$ is an increasing and $q(x)$ is a decreasing function of y for every fixed x. $(iii) p(x, 0) = q(x, 0)$ for every x and $p(0, 0) = 0$.
Show that the simultaneous equations $p(x, y) = a, q(x, y) = b$ have a unique solution in the set $x \geq 0, y \geq 0$ for all a, b satisfying $0 \leq b \leq a$ but lack a solution in the same set if $a < b$.

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15 Prove that for every positive integer n coprime to 10 there exists a multiple of n that does not contain the digit 1 in its decimal representation.

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16 Let f be an injective function from $1, 2, 3, \dots$ in itself. Prove that for any n we have: $\sum_{k=1}^{n} f(k)k^{-2} \geq \sum_{k=1}^{n} k^{-1}$.

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17 Prove that for any positive integers x, y, z with $xy - z^2 = 1$ one can find non-negative integers a, b, c, d such that $x = a^2 + b^2, y = c^2 + d^2, z = ac + bd$.
Set $z = (2q)!$ to deduce that for any prime number $p = 4q + 1$, p can be represented as the sum of squares of two integers.

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18 Given a natural number n, prove that the number $M(n)$ of points with integer coordinates inside the circle $(O(0, 0), \sqrt{n})$ satisfies

$$\pi n - 5\sqrt{n} + 1 < M(n) < \pi n + 4\sqrt{n} + 1$$

19 We consider three distinct half-lines Ox, Oy, Oz in a plane. Prove the existence and uniqueness of three points $A \in Ox, B \in Oy, C \in Oz$ such that the perimeters of the triangles OAB, OBC, OCA are all equal to a given number $2p > 0$.

20 Let O be the center of a circle. Let OU, OV be perpendicular radii of the circle. The chord PQ passes through the midpoint M of UV. Let W be a point such that $PM = PW$, where U, V, M, W are collinear. Let R be a point such that $PR = MQ$, where R lies on the line PW. Prove that $MR = UV$.

Alternative version: A circle S is given with center O and radius r. Let M be a point whose distance from O is $\frac{r}{\sqrt{2}}$. Let PMQ be a chord of S. The point N is defined by $\overrightarrow{PN} = \overrightarrow{MQ}$. Let R be the reflection of N by the line through P that is parallel to OM. Prove that $MR = \sqrt{2}r$.

21 A circle touches the sides AB, BC, CD, DA of a square at points K, L, M, N respectively, and BU, KV are parallel lines such that U is on DM and V on DN. Prove that UV touches the circle.

22 Let x and y be two integers not equal to 0 such that $x + y$ is a divisor of $x^2 + y^2$. And let $\frac{x^2+y^2}{x+y}$ be a divisor of 1978. Prove that $x = y$.

German IMO Selection Test 1979, problem 2

23 Let S be the set of all the odd positive integers that are not multiples of 5 and that are less than $30m$, m being an arbitrary positive integer. What is the smallest integer k such that in any subset of k integers from S there must be two different integers, one of which divides the other?

24 Let $0 < f(1) < f(2) < f(3) < \dots$ a sequence with all its terms positive. The $n-th$ positive integer which doesn't belong to the sequence is $f(f(n)) + 1$. Find $f(240)$.

25 Consider a polynomial $P(x) = ax^2 + bx + c$ with $a > 0$ that has two real roots x_1, x_2. Prove that the absolute values of both roots are less than or equal to 1 if and only if $a+b+c \geq 0, a-b+c \geq 0$, and $a - c \geq 0$.

26 For every integer $d \geq 1$, let M_d be the set of all positive integers that cannot be written as a sum of an arithmetic progression with difference d, having at least two terms and consisting of positive integers. Let $A = M_1, B = M_2 \setminus \{2\}, C = M_3$. Prove that every $c \in C$ may be written in a unique way as $c = ab$ with $a \in A, b \in B$.

27 Determine the sixth number after the decimal point in the number $(\sqrt{1978} + \lfloor\sqrt{1978}\rfloor)^{20}$

28 Let c, s be real functions defined on $\mathbb{R} \setminus \{0\}$ that are nonconstant on any interval and satisfy

$$c\left(\frac{x}{y}\right) = c(x)c(y) - s(x)s(y) \text{ for any } x \neq 0, y \neq 0$$

Prove that: $(a) c\left(\frac{1}{x}\right) = c(x), s\left(\frac{1}{x}\right) = -s(x)$ for any $x = 0$, and also $c(1) = 1, s(1) = s(-1) = 0$; $(b) c$ and s are either both even or both odd functions (a function f is even if $f(x) = f(-x)$ for all x, and odd if $f(x) = -f(-x)$ for all x).
Find functions c, s that also satisfy $c(x) + s(x) = x^n$ for all x, where n is a given positive integer.

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29 Given a nonconstant function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that $f(xy) = f(x)f(y)$ for any $x, y > 0$, find functions $c, s : \mathbb{R}^+ \longrightarrow \mathbb{R}$ that satisfy $c\left(\frac{x}{y}\right) = c(x)c(y) - s(x)s(y)$ for all $x, y > 0$ and $c(x) + s(x) = f(x)$ for all $x > 0$.

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30 An international society has its members from six different countries. The list of members contain 1978 names, numbered $1, 2, \dots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

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31 Let the polynomials

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$
$$Q(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

be given satisfying the identity $P(x)^2 = (x^2 - 1)Q(x)^2 + 1$. Prove the identity

$$P'(x) = nQ(x).$$

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32 Let $\mathcal{C}$ be the circumcircle of the square with vertices $(0, 0), (0, 1978), (1978, 0), (1978, 1978)$ in the Cartesian plane. Prove that $\mathcal{C}$ contains no other point for which both coordinates are integers.

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33 A sequence $(a_n)_0^\infty$ of real numbers is called *convex* if $2a_n \leq a_{n-1} + a_{n+1}$ for all positive integers n. Let $(b_n)_0^\infty$ be a sequence of positive numbers and assume that the sequence $(\alpha^n b_n)_0^\infty$ is convex for any choice of $\alpha > 0$. Prove that the sequence $(\log b_n)_0^\infty$ is convex.

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34 A function $f : I \to \mathbb{R}$, defined on an interval I, is called concave if $f(\theta x + (1 - \theta)y) \geq \theta f(x) + (1 - \theta)f(y)$ for all $x, y \in I$ and $0 \leq \theta \leq 1$. Assume that the functions $f_1, \dots, f_n$, having all nonnegative values, are concave. Prove that the function $(f_1 f_2 \cdots f_n)^{1/n}$ is concave.

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35 A sequence $(a_n)_0^N$ of real numbers is called concave if $2a_n \geq a_{n-1} + a_{n+1}$ for all integers $n, 1 \leq n \leq N-1$. (a) Prove that there exists a constant $C > 0$ such that

$$\left(\sum_{n=0}^{N} a_n\right)^2 \geq C(N-1)\sum_{n=0}^{N} a_n^2 \quad (1)$$

for all concave positive sequences $(a_n)_0^N$ (b) Prove that (1) holds with $C = \frac{3}{4}$ and that this constant is best
possible.

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36 The integers 1 through 1000 are located on the circumference of a circle in natural order. Starting with 1, every fifteenth number (i.e.,$1, 16, 31, \cdots$) is marked. The marking is continued until an already marked number is reached. How many of the numbers will be left unmarked?

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37 Simplify

$$\frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)},$$

where a, b, c are positive real numbers.

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38 Given a circle, construct a chord that is trisected by two given noncollinear radii.

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39 A is a $2m$-digit positive integer each of whose digits is 1. B is an m-digit positive integer each of whose digits is 4. Prove that $A + B + 1$ is a perfect square.

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40 If $C_n^p = \frac{n!}{p!(n-p)!}(p \geq 1)$, prove the identity

$$C_n^p = C_{n-1}^{p-1} + C_{n-2}^{p-1} + \cdots + C_p^{p-1} + C_{p-1}^{p-1}$$

and then evaluate the sum

$$S = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + 97 \cdot 98 \cdot 99.$$

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41 In a triangle ABC we have $AB = AC$. A circle which is internally tangent with the circumscribed circle of the triangle is also tangent to the sides AB, AC in the points P, respectively Q. Prove that the midpoint of PQ is the center of the inscribed circle of the triangle ABC.

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42 A, B, C, D, E are points on a circle O with radius equal to r. Chords AB and DE are parallel to each other and have length equal to x. Diagonals AC, AD, BE, CE are drawn. If segment XY on O meets AC at X and EC at Y , prove that lines BX and DY meet at Z on the circle.

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43 If p is a prime greater than 3, show that at least one of the numbers

$$\frac{3}{p^2}, \frac{4}{p^2}, \dots, \frac{p-2}{p^2}$$

is expressible in the form $\frac{1}{x}+\frac{1}{y}$, where x and y are positive integers.

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44 In ABC with $\angle C = 60^\circ$, prove that

$$\frac{c}{a}+\frac{c}{b} \ge 2.$$

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45 If $r > s > 0$ and $a > b > c$, prove that

$$a^r b^s + b^r c^s + c^r a^s \ge a^s b^r + b^s c^r + c^s a^r.$$

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46 We consider a fixed point P in the interior of a fixed sphere. We construct three segments PA, PB, PC, perpendicular two by two, with the vertexes A, B, C on the sphere. We consider the vertex Q which is opposite to P in the parallelepiped (with right angles) with PA, PB, PC as edges. Find the locus of the point Q when A, B, C take all the positions compatible with our problem.

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47 Given the expression

$$P_n(x) = \frac{1}{2^n}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right],$$

prove: $(a) P_n(x)$ satisfies the identity

$$P_n(x) - xP_{n-1}(x) + \frac{1}{4}P_{n-2}(x) \equiv 0.$$

$(b) P_n(x)$ is a polynomial in x of degree n.

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48 Prove that it is possible to place $2n(2n+1)$ parallelepipedic (rectangular) pieces of soap of dimensions $1 \times 2 \times (n+1)$ in a cubic box with edge $2n+1$ if and only if n is even or $n = 1$.

Remark. It is assumed that the edges of the pieces of soap are parallel to the edges of the box.

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49 Let A, B, C, D be four arbitrary distinct points in space. (a) Prove that using the segments $AB + CD$, $AC + BD$ and $AD + BC$, it is always possible to construct a triangle T that is non-degenerate and has no obtuse angle. (b) What should these four points satisfy in order for the triangle T to be right-angled?

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50 A variable tetrahedron $ABCD$ has the following properties:
Its edge lengths can change as well as its vertices, but the opposite edges remain equal $(BC = DA, CA = DB, AB = DC)$; and the vertices A, B, C lie respectively on three fixed spheres with the same center P and radii $3, 4, 12$. What is the maximal length of PD?

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51 Find the relations among the angles of the triangle ABC whose altitude AH and median AM satisfy $\angle BAH = \angle CAM$.

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52 Let p be a prime and $A = \{a_1, \dots, a_{p-1}\}$ an arbitrary subset of the set of natural numbers such that none of its elements is divisible by p. Let us define a mapping f from $\mathcal{P}(A)$ (the set of all subsets of A) to the set $P = \{0, 1, \dots, p-1\}$ in the following way:

(i) if $B = \{a_{i_1}, \dots, a_{i_k}\} \subset A$ and $\sum_{j=1}^{k} a_{i_j} \equiv n \pmod p$, then $f(B) = n$,

(ii) $f(\emptyset) = 0$, $\emptyset$ being the empty set.

Prove that for each $n \in P$ there exists $B \subset A$ such that $f(B) = n$.

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53 Determine all the triples (a, b, c) of positive real numbers such that the system

$$ax + by - cz = 0,$$

$$a\sqrt{1-x^2} + b\sqrt{1-y^2} - c\sqrt{1-z^2} = 0,$$

is compatible in the set of real numbers, and then find all its real solutions.

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54 Let p, q and r be three lines in space such that there is no plane that is parallel to all three of them. Prove that there exist three planes α, β, and γ, containing p, q, and r respectively, that are perpendicular to each other $(\alpha \perp \beta, \beta \perp \gamma, \gamma \perp \alpha)$.
