IMO Longlists 1982

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1 **(a)** Prove that $\frac{1}{n+1} \cdot \binom{2n}{n}$ is an integer for $n \geq 0$.

(b) Given a positive integer k, determine the smallest integer C_k with the property that $\frac{C_k}{n+k+1} \cdot \binom{2n}{n}$ is an integer for all $n \geq k$.

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3 Given n points $X_1, X_2, \ldots, X_n$ in the interval $0 \leq X_i \leq 1, i = 1, 2, \ldots, n$, show that there is a point $y, 0 \leq y \leq 1$, such that

$$\frac{1}{n} \sum_{i=1}^{n} |y - X_i| = \frac{1}{2}.$$

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4 **(a)** Find the rearrangement $\{a_1, \ldots, a_n\}$ of $\{1, 2, \ldots, n\}$ that maximizes

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 = Q.$$

(b) Find the rearrangement that minimizes Q.

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5 Among all triangles with a given perimeter, find the one with the maximal radius of its incircle.

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6 On the three distinct lines a, b, and c three points A, B, and C are given, respectively. Construct three collinear points X, Y, Z on lines a, b, c, respectively, such that $\frac{BY}{AX} = 2$ and $\frac{CZ}{AX} = 3$.

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7 Find all solutions $(x, y) \in \mathbb{Z}^2$ of the equation

$$x^3 - y^3 = 2xy + 8.$$

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8 A box contains p white balls and q black balls. Beside the box there is a pile of black balls. Two balls are taken out of the box. If they have the same color, a black ball from the pile is put into the box. If they have different colors, the white ball is put back into the box. This procedure is repeated until the last two balls are removed from the box and one last ball is put in. What is the probability that this last ball is white?

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9 Given any two real numbers α and $\beta, 0 \leq \alpha < \beta \leq 1$, prove that there exists a natural number m such that

$$\alpha < \frac{\phi(m)}{m} < \beta.$$

10 Let $r_1, \dots, r_n$ be the radii of n spheres. Call $S_1, S_2, \dots, S_n$ the areas of the set of points of each sphere from which one cannot see any point of any other sphere. Prove that

$$\frac{S_1}{r_1^2} + \frac{S_2}{r_2^2} + \cdots + \frac{S_n}{r_n^2} = 4\pi.$$

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11 A rectangular pool table has a hole at each of three of its corners. The lengths of sides of the table are the real numbers a and b. A billiard ball is shot from the fourth corner along its angle bisector. The ball falls in one of the holes. What should the relation between a and b be for this to happen?

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12 Let there be 3399 numbers arbitrarily chosen among the first 6798 integers $1, 2, \dots, 6798$ in such a way that none of them divides another. Prove that there are exactly 1982 numbers in $\{1, 2, \dots, 6798\}$ that must end up being chosen.

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13 A regular n-gonal truncated pyramid is circumscribed around a sphere. Denote the areas of the base and the lateral surfaces of the pyramid by S_1, S_2, and S, respectively. Let σ be the area of the polygon whose vertices are the tangential points of the sphere and the lateral faces of the pyramid. Prove that

$$\sigma S = 4S_1S_2\cos^2\frac{\pi}{n}.$$

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14 Determine all real values of the parameter a for which the equation

$$16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$$

has exactly four distinct real roots that form a geometric progression.

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15 Show that the set S of natural numbers n for which $\frac{3}{n}$ cannot be written as the sum of two reciprocals of natural numbers ($S = \left\{n | \frac{3}{n} \neq \frac{1}{p} + \frac{1}{q} \text{ for any } p, q \in \mathbb{N}\right\}$) is not the union of finitely many arithmetic progressions.

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16 Let $p(x)$ be a cubic polynomial with integer coefficients with leading coefficient 1 and with one of its roots equal to the product of the other two. Show that $2p(-1)$ is a multiple of $p(1) + p(-1) - 2(1 + p(0))$.

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17 **(a)** Find the rearrangement $\{a_1, \dots, a_n\}$ of $\{1, 2, \dots, n\}$ that maximizes

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 = Q.$$

(b) Find the rearrangement that minimizes Q.

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18 You are given an algebraic system admitting addition and multiplication for which all the laws of ordinary arithmetic are valid except commutativity of multiplication. Show that

$$(a + ab^{-1}a)^{-1} + (a + b)^{-1} = a^{-1},$$

where x^{-1} is the element for which $x^{-1}x = xx^{-1} = e$, where e is the element of the system such that for all a the equality $ea = ae = a$ holds.

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19 Show that

$$\frac{1 - s^a}{1 - s} \leq (1 + s)^{a-1}$$

holds for every $1 \neq s > 0$ real and $0 < a \leq 1$ rational.

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20 Consider a cube C and two planes σ, τ, which divide Euclidean space into several regions. Prove that the interior of at least one of these regions meets at least three faces of the cube.

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21 All edges and all diagonals of regular hexagon $A_1A_2A_3A_4A_5A_6$ are colored blue or red such that each triangle $A_jA_kA_m, 1 \leq j < k < m \leq 6$ has at least one red edge. Let R_k be the number of red segments $A_kA_j, (j \neq k)$. Prove the inequality

$$\sum_{k=1}^{6} (2R_k - 7)^2 \leq 54.$$

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22 Let M be the set of real numbers of the form $\frac{m+n}{\sqrt{m^2+n^2}}$, where m and n are positive integers. Prove that for every pair $x \in M, y \in M$ with $x < y$, there exists an element $z \in M$ such that $x < z < y$.

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23 Determine the sum of all positive integers whose digits (in base ten) form either a strictly increasing or a strictly decreasing sequence.

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24 Prove that if a person a has infinitely many descendants (children, their children, etc.), then a has an infinite sequence $a_0, a_1, \ldots$ of descendants (i.e., $a = a_0$ and for all $n \geq 1, a_{n+1}$ is always a child of a_n). It is assumed that no-one can have infinitely many children.

Variant 1. Prove that if a has infinitely many ancestors, then a has an infinite descending sequence of ancestors (i.e., $a_0, a_1, \ldots$ where $a = a_0$ and a_n is always a child of a_{n+1}).

Variant 2. Prove that if someone has infinitely many ancestors, then all people cannot descend from $A(dam)$ and $E(ve)$.

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25 Four distinct circles C, C_1, C_2, C3 and a line L are given in the plane such that C and L are disjoint and each of the circles C_1, C_2, C_3 touches the other two, as well as C and L. Assuming the radius of C to be 1, determine the distance between its center and L.

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26 Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of natural numbers. Determine whether there exists a pair (p, q) of natural numbers that satisfy

$$p < q \quad \text{and} \quad a_p \leq a_q, b_p \leq b_q.$$

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27 Let O be a point of three-dimensional space and let l_1, l_2, l_3 be mutually perpendicular straight lines passing through O. Let S denote the sphere with center O and radius R, and for every point M of S, let S_M denote the sphere with center M and radius R. We denote by P_1, P_2, P_3 the intersection of S_M with the straight lines l_1, l_2, l_3, respectively, where we put $P_i \neq O$ if l_i meets S_M at two distinct points and $P_i = O$ otherwise ($i = 1, 2, 3$). What is the set of centers of gravity of the (possibly degenerate) triangles $P_1P_2P_3$ as M runs through the points of S?

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28 Let $(u_1, \ldots, u_n)$ be an ordered $n$tuple. For each $k, 1 \leq k \leq n$, define $v_k = \sqrt[k]{u_1u_2\cdots u_k}$. Prove that

$$\sum_{k=1}^{n} v_k \leq e \cdot \sum_{k=1}^{n} u_k.$$

(e is the base of the natural logarithm).

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29 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that the restriction of f to the set of irrational numbers is injective. What can we say about f? Answer the analogous question if f is restricted to rationals.

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30 Let ABC be a triangle, and let P be a point inside it such that $\angle PAC = \angle PBC$. The perpendiculars from P to BC and CA meet these lines at L and M, respectively, and D is the midpoint of AB. Prove that $DL = DM$.

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31 Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers x, y, then it has at least three such solutions. Show that the equation has no solutions in integers for $n = 2891$.

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32 The function $f(n)$ is defined on the positive integers and takes non-negative integer values. $f(2) = 0, f(3) > 0, f(9999) = 3333$ and for all $m, n :$

$$f(m + n) - f(m) - f(n) = 0 \text{ or } 1.$$

Determine $f(1982)$.

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34 Let M be the set of all functions f with the following properties:

(i) f is defined for all real numbers and takes only real values.

(ii) For all $x, y \in \mathbb{R}$ the following equality holds: $f(x)f(y) = f(x+y) + f(x-y)$.

(iii) $f(0) \neq 0$.

Determine all functions $f \in M$ such that

(a) $f(1) = \frac{5}{2}$,

(b) $f(1) = \sqrt{3}$.

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35 If the inradius of a triangle is half of its circumradius, prove that the triangle is equilateral.

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36 A non-isosceles triangle $A_1A_2A_3$ has sides a_1, a_2, a_3 with the side a_i lying opposite to the vertex A_i. Let M_i be the midpoint of the side a_i, and let T_i be the point where the inscribed circle of triangle $A_1A_2A_3$ touches the side a_i. Denote by S_i the reflection of the point T_i in the interior angle bisector of the angle A_i. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.

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37 The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by inner points M and N respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M and N are collinear.

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38 Numbers $u_{n,k}$ $(1 \leq k \leq n)$ are defined as follows

$$u_{1,1} = 1, \quad u_{n,k} = \binom{n}{k} - \sum_{d|n, d|k, d>1} u_{n/d, k/d}.$$

(the empty sum is defined to be equal to zero). Prove that $n \mid u_{n,k}$ for every natural number n and for every k $(1 \leq k \leq n)$.

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39 Let S be the unit circle with center O and let $P_1, P_2, \dots, P_n$ be points of S such that the sum of vectors $v_i = \overrightarrow{OP_i}$ is the zero vector. Prove that the inequality $\sum_{i=1}^{n} XP_i \geq n$ holds for every point X.

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40 We consider a game on an infinite chessboard similar to that of solitaire: If two adjacent fields are occupied by pawns and the next field is empty (the three fields lie on a vertical or horizontal line), then we may remove these two pawns and put one of them on the third field. Prove that if in the initial position pawns fill a $3k \times n$ rectangle, then it is impossible to reach a position with only one pawn on the board.

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41 A convex, closed figure lies inside a given circle. The figure is seen from every point of the circumference at a right angle (that is, the two rays drawn from the point and supporting the convex figure are perpendicular). Prove that the center of the circle is a center of symmetry of the figure.

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42 Let $\mathfrak{F}$ be the family of all k-element subsets of the set $\{1, 2, \dots, 2k+1\}$. Prove that there exists a bijective function $f : \mathfrak{F} \to \mathfrak{F}$ such that for every $A \in \mathfrak{F}$, the sets A and $f(A)$ are disjoint.

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43 **(a)** What is the maximal number of acute angles in a convex polygon?

(b) Consider m points in the interior of a convex n-gon. The n-gon is partitioned into triangles whose vertices are among the $n+m$ given points (the vertices of the n-gon and the given points). Each of the m points in the interior is a vertex of at least one triangle. Find the number of triangles obtained.

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44 Let A and B be positions of two ships M and N, respectively, at the moment when N saw M moving with constant speed v following the line Ax. In search of help, N moves with speed kv ($k < 1$) along the line By in order to meet M as soon as possible. Denote by C the point of meeting of the two ships, and set

$$AB = d, \angle BAC = \alpha, 0 \leq \alpha < \frac{\pi}{2}.$$

Determine the angle $\angle ABC = \beta$ and time t that N needs in order to meet M.

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45 Let $ABCD$ be a convex quadrilateral and draw regular triangles ABM, CDP, BCN, ADQ, the first two outward and the other two inward. Prove that $MN = AC$. What can be said about the quadrilateral $MNPQ$?

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46 Prove that if a diagonal is drawn in a quadrilateral inscribed in a circle, the sum of the radii of the circles inscribed in the two triangles thus formed is the same, no matter which diagonal is drawn.

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47 Evaluate $\sec'' \frac{\pi}{4} + \sec'' \frac{3\pi}{4} + \sec'' \frac{5\pi}{4} + \sec'' \frac{7\pi}{4}$. (Here $\sec''$ means the second derivative of $\sec$).

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48 Given a finite sequence of complex numbers $c_1, c_2, \dots, c_n$, show that there exists an integer k ($1 \leq k \leq n$) such that for every finite sequence $a_1, a_2, \dots, a_n$ of real numbers with $1 \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, the following inequality holds:

$$\left| \sum_{m=1}^{n} a_m c_m \right| \leq \left| \sum_{m=1}^{k} c_m \right|.$$

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49 Simplify

$$\sum_{k=0}^{n} \frac{(2n)!}{(k!)^2((n-k)!)^2}.$$

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50 Let O be the midpoint of the axis of a right circular cylinder. Let A and B be diametrically opposite points of one base, and C a point of the other base circle that does not belong to the plane OAB. Prove that the sum of dihedral angles of the trihedral $OABC$ is equal to 2π.

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51 Let n numbers $x_1, x_2, \dots, x_n$ be chosen in such a way that $1 \geq x_1 \geq x_2 \geq \dots \geq x_n \geq 0$. Prove that

$$(1 + x_1 + x_2 + \dots + x_n)^\alpha \leq 1 + x_1^\alpha + 2^{\alpha-1}x_2^\alpha + \dots + n^{\alpha-1}x_n^\alpha$$

if $0 \leq \alpha \leq 1$.

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52 We are given $2n$ natural numbers

$$1, 1, 2, 2, 3, 3, \dots, n-1, n-1, n, n.$$

Find all n for which these numbers can be arranged in a row such that for each $k \leq n$, there are exactly k numbers between the two numbers k.

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53 Consider infinite sequences $\{x_n\}$ of positive reals such that $x_0 = 1$ and $x_0 \geq x_1 \geq x_2 \geq \dots$.

a) Prove that for every such sequence there is an $n \geq 1$ such that:

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

b) Find such a sequence such that for all n:

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} < 4.$$

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54 The right triangles ABC and AB_1C_1 are similar and have opposite orientation. The right angles are at C and C_1, and we also have $\angle CAB = \angle C_1AB_1$. Let M be the point of intersection of the lines BC_1 and B_1C. Prove that if the lines AM and CC_1 exist, they are perpendicular.

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55 Let S be a square with sides length 100. Let L be a path within S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, A_2A_3, \dots, A_{n-1}A_n$ with $A_0 = A_n$. Suppose that for every point P on the boundary of S there is a point of L at a distance from P no greater than $\frac{1}{2}$. Prove that there are two points X and Y of L such that the distance between X and Y is not greater than 1 and the length of the part of L which lies between X and Y is not smaller than 198.

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56 Let $f(x) = ax^2 + bx + c$ and $g(x) = cx^2 + bx + a$. If $|f(0)| \leq 1, |f(1)| \leq 1, |f(-1)| \leq 1$, prove that for $|x| \leq 1$,

(a) $|f(x)| \leq 5/4$,

(b) $|g(x)| \leq 2$.

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57 Let K be a convex polygon in the plane and suppose that K is positioned in the coordinate system in such a way that

$$\text{area } (K \cap Q_i) = \frac{1}{4}\text{area } K \ (i = 1, 2, 3, 4,),$$

where the Q_i denote the quadrants of the plane. Prove that if K contains no nonzero lattice point, then the area of K is less than 4.
