IMO Longlists 1986

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1 Let k be one of the integers $2, 3, 4$ and let $n = 2^k - 1$. Prove the inequality

$$1 + b^k + b^{2k} + \cdots + b^{nk} \geq (1 + b^n)^k$$

for all real $b \geq 0$.

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2 Let $ABCD$ be a convex quadrilateral. DA and CB meet at F and AB and DC meet at E. The bisectors of the angles DFC and AED are perpendicular. Prove that these angle bisectors are parallel to the bisectors of the angles between the lines AC and BD.

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3 A line parallel to the side BC of a triangle ABC meets AB in F and AC in E. Prove that the circles on BE and CF as diameters intersect in a point lying on the altitude of the triangle ABC dropped from A to BC.

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4 Find the last eight digits of the binary development of 27^{1986}.

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5 Let ABC and DEF be acute-angled triangles. Write $d = EF, e = FD, f = DE$. Show that there exists a point P in the interior of ABC for which the value of the expression $X = d \cdot AP + e \cdot BP + f \cdot CP$ attains a minimum.

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6 In an urn there are one ball marked 1, two balls marked 2, and so on, up to n balls marked n. Two balls are randomly drawn without replacement. Find the probability that the two balls are assigned the same number.

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7 Let $f(n)$ be the least number of distinct points in the plane such that for each $k = 1, 2, \cdots, n$ there exists a straight line containing exactly k of these points. Find an explicit expression for $f(n)$.

Simplified version.

Show that $f(n) = \left[\frac{n+1}{2}\right] \left[\frac{n+2}{2}\right]$. Where $[x]$ denoting the greatest integer not exceeding x.

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8 A tetrahedron $ABCD$ is given such that $AD = BC = a; AC = BD = b; AB \cdot CD = c^2$. Let $f(P) = AP + BP + CP + DP$, where P is an arbitrary point in space. Compute the least value of $f(P)$.

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9 In a triangle ABC, $\angle BAC = 100^\circ, AB = AC$. A point D is chosen on the side AC such that $\angle ABD = \angle CBD$. Prove that $AD + DB = BC$.

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10 A set of n standard dice are shaken and randomly placed in a straight line. If $n < 2r$ and $r < s$, then the probability that there will be a string of at least r, but not more than s, consecutive 1's can be written as $\frac{P}{6^{s+2}}$. Find an explicit expression for P.

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11 Prove that the sum of the face angles at each vertex of a tetrahedron is a straight angle if and only if the faces are congruent triangles.

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12 Let O be an interior point of a tetrahedron $A_1A_2A_3A_4$. Let S_1, S_2, S_3, S_4 be spheres with centers A_1, A_2, A_3, A_4, respectively, and let U, V be spheres with centers at O. Suppose that for $i, j = 1, 2, 3, 4, i \neq j$, the spheres S_i and S_j are tangent to each other at a point B_{ij} lying on A_iA_j. Suppose also that U is tangent to all edges A_iA_j and V is tangent to the spheres S_1, S_2, S_3, S_4. Prove that $A_1A_2A_3A_4$ is a regular tetrahedron.

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13 Let $N = \{1, 2, \dots, n\}, n \geq 3$. To each pair $i \neq j$ of elements of N there is assigned a number $f_{ij} \in \{0, 1\}$ such that $f_{ij} + f_{ji} = 1$.
Let $r(i) = \sum_{i \neq j} f_{ij}$, and write $M = \max_{i \in N} r(i)$, $m = \min_{i \in N} r(i)$. Prove that for any $w \in N$ with $r(w) = m$ there exist $u, v \in N$ such that $r(u) = M$ and $f_{uv}f_{vw} = 1$.

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14 Given a point P_0 in the plane of the triangle $A_1A_2A_3$. Define $A_s = A_{s-3}$ for all $s \geq 4$. Construct a set of points $P_1, P_2, P_3, \dots$ such that P_{k+1} is the image of P_k under a rotation center A_{k+1} through an angle 120^o clockwise for $k = 0, 1, 2, \dots$. Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.

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15 Let $\mathbb{N} = B_1 \cup \dots \cup B_q$ be a partition of the set $\mathbb{N}$ of all positive integers and let an integer $l \in \mathbb{N}$ be given. Prove that there exist a set $X \subset \mathbb{N}$ of cardinality l, an infinite set $T \subset \mathbb{N}$, and an integer k with $1 \leq k \leq q$ such that for any $t \in T$ and any finite set $Y \subset X$, the sum $t + \sum_{y \in Y} y$ belongs to B_k.

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16 Given a positive integer k, find the least integer n_k for which there exist five sets S_1, S_2, S_3, S_4, S_5 with the following properties:

$$|S_j| = k \text{ for } j = 1, \cdots, 5, \quad |\bigcup_{j=1}^{5} S_j| = n_k;$$

$$|S_i \cap S_{i+1}| = 0 = |S_5 \cap S_1|, \quad \text{for } i = 1, \cdots, 4$$

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17 We call a tetrahedron right-faced if each of its faces is a right-angled triangle.

(a) Prove that every orthogonal parallelepiped can be partitioned into six right-faced tetrahedra.

(b) Prove that a tetrahedron with vertices A_1, A_2, A_3, A_4 is right-faced if and only if there exist four distinct real numbers c_1, c_2, c_3, and c_4 such that the edges A_jA_k have lengths $A_jA_k = \sqrt{|c_j - c_k|}$ for $1 \leq j < k \leq 4$.

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18 Provided the equation $xyz = p^n(x + y + z)$ where $p \geq 3$ is a prime and $n \in \mathbb{N}$. Prove that the equation has at least $3n + 3$ different solutions (x, y, z) with natural numbers x, y, z and $x < y < z$. Prove the same for $p > 3$ being an odd integer.

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19 Let $f : [0, 1] \to [0, 1]$ satisfy $f(0) = 0, f(1) = 1$ and

$$f(x + y) - f(x) = f(x) - f(x - y)$$

for all $x, y \geq 0$ with $x - y, x + y \in [0, 1]$. Prove that $f(x) = x$ for all $x \in [0, 1]$.

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20 For any angle with $0 < \alpha < 180^\circ$, we call a closed convex planar set an α-set if it is bounded by two circular arcs (or an arc and a line segment) whose angle of intersection is α. Given a (closed) triangle T , find the greatest α such that any two points in T are contained in an α-set $S \subset T$.

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21 Let AB be a segment of unit length and let C, D be variable points of this segment. Find the maximum value of the product of the lengths of the six distinct segments with endpoints in the set $\{A, B, C, D\}$.

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22 Let $(a_n)_{n\geq 0}$ be the sequence of integers defined recursively by $a_0 = 0, a_1 = 1, a_{n+2} = 4a_{n+1}+a_n$ for $n \geq 0$. Find the common divisors of a_{1986} and a_{6891}.

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23 Let I and J be the centers of the incircle and the excircle in the angle BAC of the triangle ABC. For any point M in the plane of the triangle, not on the line BC, denote by I_M and J_M the centers of the incircle and the excircle (touching BC) of the triangle BCM. Find the locus of points M for which II_MJJ_M is a rectangle.

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24 Two families of parallel lines are given in the plane, consisting of 15 and 11 lines, respectively. In each family, any two neighboring lines are at a unit distance from one another; the lines of the first family are perpendicular to the lines of the second family. Let V be the set of 165 intersection points of the lines under consideration. Show that there exist not fewer than 1986 distinct squares with vertices in the set V.

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25 Let real numbers $x_1, x_2, \cdots, x_n$ satisfy $0 < x_1 < x_2 < \cdots < x_n < 1$ and set $x_0 = 0, x_{n+1} = 1$. Suppose that these numbers satisfy the following system of equations:

$$\sum_{j=0, j\neq i}^{n+1} \frac{1}{x_i - x_j} = 0 \quad \text{where } i = 1, 2, ..., n.$$

Prove that $x_{n+1-i} = 1 - x_i$ for $i = 1, 2, ..., n$.

26 Let d be any positive integer not equal to $2, 5$ or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

27 In an urn there are n balls numbered $1, 2, \cdots, n$. They are drawn at random one by one without replacement and the numbers are recorded. What is the probability that the resulting random permutation has only one local maximum?
A term in a sequence is a local maximum if it is greater than all its neighbors.

28 A particle moves from $(0, 0)$ to (n, n) directed by a fair coin. For each head it moves one step east and for each tail it moves one step north. At $(n, y), y < n$, it stays there if a head comes up and at $(x, n), x < n$, it stays there if a tail comes up. Let$k$ be a fixed positive integer. Find the probability that the particle needs exactly $2n + k$ tosses to reach (n, n).

29 We define a binary operation $\star$ in the plane as follows: Given two points A and B in the plane, $C = A \star B$ is the third vertex of the equilateral triangle ABC oriented positively. What is the relative position of three points I, M, O in the plane if $I \star (M \star O) = (O \star I) \star M$ holds?

30 Prove that a convex polyhedron all of whose faces are equilateral triangles has at most 30 edges.

31 Let P and Q be distinct points in the plane of a triangle ABC such that $AP : AQ = BP : BQ = CP : CQ$. Prove that the line PQ passes through the circumcenter of the triangle.

32 Find, with proof, all solutions of the equation $\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1$ in positive integers x, y, z.

33 Let A, B be adjacent vertices of a regular n-gon ($n \geq 5$) with center O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, with X remaining inside the polygon. Find the locus of X.

34 For each non-negative integer n, $F_n(x)$ is a polynomial in x of degree n. Prove that if the identity

$$F_n(2x) = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} 2^r F_r(x)$$

holds for each n, then

$$F_n(tx) = \sum_{r=0}^{n} \binom{n}{r} t^r (1-t)^{n-r} F_r(x)$$

35 Establish the maximum and minimum values that the sum $|a| + |b| + |c|$ can have if a, b, c are real numbers such that the maximum value of $|ax^2 + bx + c|$ is 1 for $-1 \leq x \leq 1$.

36 Given a finite set of points in the plane, each with integer coordinates, is it always possible to color the points red or white so that for any straight line L parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on L is not greater than 1?

37 Prove that the set $\{1, 2, ..., 1986\}$ can be partitioned into 27 disjoint sets so that no one of these sets contains an arithmetic triple (i.e., three distinct numbers in an arithmetic progression).

38 To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

39 Let S be a k-element set.

(a) Find the number of mappings $f : S \to S$ such that

(i) $f(x) \neq x$ for $x \in S$, (ii) $f(f(x)) = x$ for $x \in S$.

(b) The same with the condition (i) left out.

40 Find the maximum value that the quantity $2m + 7n$ can have such that there exist distinct positive integers x_i $(1 \leq i \leq m), y_j$ $(1 \leq j \leq n)$ such that the x_i's are even, the y_j's are odd, and $\sum_{i=1}^m x_i + \sum_{j=1}^n y_j = 1986$.

41 Let M, N, P be the midpoints of the sides BC, CA, AB of a triangle ABC. The lines AM, BN, CP intersect the circumcircle of ABC at points A', B', C', respectively. Show that if $A'B'C'$ is an equilateral triangle, then so is ABC.

42 The integers $1, 2, \cdots, n^2$ are placed on the fields of an $n \times n$ chessboard $(n > 2)$ in such a way that any two fields that have a common edge or a vertex are assigned numbers differing by at most $n + 1$. What is the total number of such placements?

43 Three persons A, B, C, are playing the following game:

A k-element subset of the set $\{1, ..., 1986\}$ is randomly chosen, with an equal probability of each choice, where k is a fixed positive integer less than or equal to 1986. The winner is A, B or C, respectively, if the sum of the chosen numbers leaves a remainder of $0, 1$, or 2 when divided by 3.

For what values of k is this game a fair one? (A game is fair if the three outcomes are equally probable.)

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44 The circle inscribed in a triangle ABC touches the sides BC, CA, AB in D, E, F, respectively, and X, Y, Z are the midpoints of EF, FD, DE, respectively. Prove that the centers of the inscribed circle and of the circles around XYZ and ABC are collinear.

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45 Given n real numbers $a_1 \leq a_2 \leq \cdots \leq a_n$, define

$$M_1 = \frac{1}{n}\sum_{i=1}^{n} a_i, \quad M_2 = \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} a_i a_j, \quad Q = \sqrt{M_1^2 - M_2}$$

Prove that

$$a_1 \leq M_1 - Q \leq M_1 + Q \leq a_n$$

and that equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

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46 We wish to construct a matrix with 19 rows and 86 columns, with entries $x_{ij} \in \{0, 1, 2\}$ $(1 \leq i \leq 19, 1 \leq j \leq 86)$, such that:

(i) in each column there are exactly k terms equal to 0;

(ii) for any distinct $j, k \in \{1, ..., 86\}$ there is $i \in \{1, ..., 19\}$ with $x_{ij} + x_{ik} = 3$.

For what values of k is this possible?

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47 Let A, B be adjacent vertices of a regular n-gon ($n \geq 5$) with center O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, with X remaining inside the polygon. Find the locus of X.

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48 Let P be a convex 1986-gon in the plane. Let A, D be interior points of two distinct sides of P and let B, C be two distinct interior points of the line segment AD. Starting with an arbitrary point Q_1 on the boundary of P, define recursively a sequence of points Q_n as follows:
given Q_n extend the directed line segment Q_nB to meet the boundary of P in a point R_n and then extend R_nC to meet the boundary of P again in a point, which is defined to be Q_{n+1}. Prove that for all n large enough the points Q_n are on one of the sides of P containing A or D.

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49 Let C_1, C_2 be circles of radius $1/2$ tangent to each other and both tangent internally to a circle C of radius 1. The circles C_1 and C_2 are the first two terms of an infinite sequence of distinct circles C_n defined as follows: C_{n+2} is tangent externally to C_n and C_{n+1} and internally to C. Show that the radius of each C_n is the reciprocal of an integer.

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50 Let D be the point on the side BC of the triangle ABC such that AD is the bisector of $\angle CAB$. Let I be the incenter ofABC.

(a) Construct the points P and Q on the sides AB and AC, respectively, such that PQ is parallel to BC and the perimeter of the triangle APQ is equal to $k \cdot BC$, where k is a given rational number.

(b) Let R be the intersection point of PQ and AD. For what value of k does the equality $AR = RI$ hold?

(c) In which case do the equalities $AR = RI = ID$ hold?

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51 Let a, b, c, d be the lengths of the sides of a quadrilateral circumscribed about a circle and let S be its area. Prove that $S \leq \sqrt{abcd}$ and find conditions for equality.

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52 Solve the system of equations

$$\tan x_1 + \cot x_1 = 3 \tan x_2,$$
$$\tan x_2 + \cot x_2 = 3 \tan x_3,$$
$$\vdots$$
$$\tan x_n + \cot x_n = 3 \tan x_1$$

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53 For given positive integers r, v, n let $S(r, v, n)$ denote the number of n-tuples of non-negative integers $(x_1, \cdots, x_n)$ satisfying the equation $x_1 + \cdots + x_n = r$ and such that $x_i \leq v$ for $i = 1, \cdots, n$. Prove that

$$S(r, v, n) = \sum_{k=0}^{m} (-1)^k \binom{n}{k} \binom{r - (v+1)k + n - 1}{n - 1}$$

Where $m = \left\{ n, \left[\frac{r}{v+1} \right] \right\}$.

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54 Find the least integer n with the following property:
For any set V of 8 points in the plane, no three lying on a line, and for any set E of n line segments with endpoints in V , one can find a straight line intersecting at least 4 segments in E in interior points.

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55 Given an integer $n \geq 2$, determine all n-digit numbers $M_0 = \overline{a_1 a_2 \cdots a_n}$ $(a_i \neq 0, i = 1, 2, ..., n)$ divisible by the numbers $M_1 = \overline{a_2 a_3 \cdots a_n a_1}$, $M_2 = \overline{a_3 a_4 \cdots a_n a_1 a_2}$, $\cdots$, $M_{n-1} = \overline{a_n a_1 a_2 ... a_{n-1}}$.

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56 Let $A_1A_2A_3A_4A_5A_6$ be a hexagon inscribed into a circle with center O. Consider the circular arc with endpoints A_1, A_6 not containing A_2. For any point M of that arc denote by h_i the distance from M to the line A_iA_{i+1} $(1 \leq i \leq 5)$. Construct M such that the sum $h_1 + \cdots + h_5$ is maximal.

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57 In a triangle ABC, the incircle touches the sides BC, CA, AB in the points A', B', C', respectively; the excircle in the angle A touches the lines containing these sides in A_1, B_1, C_1, and similarly, the excircles in the angles B and C touch these lines in A_2, B_2, C_2 and A_3, B_3, C_3. Prove that the triangle ABC is right-angled if and only if one of the point triples (A', B_3, C'), $(A_3, B', C_3), (A', B', C_2), (A_2, B_2, C'), (A_2, B_1, C_2), (A_3, B_3, C_1), (A_1, B_2, C_1), (A_1, B_1, C_3)$ is collinear.

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58 Find four positive integers each not exceeding 70000 and each having more than 100 divisors.

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59 Let $ABCD$ be a convex quadrilateral whose vertices do not lie on a circle. Let $A'B'C'D'$ be a quadrangle such that A', B', C', D' are the centers of the circumcircles of triangles BCD, ACD, ABD, and ABC. We write $T(ABCD) = A'B'C'D'$. Let us define $A''B''C''D'' = T(A'B'C'D') = T(T(ABCD))$.

(a) Prove that $ABCD$ and $A''B''C''D''$ are similar.

(b) The ratio of similitude depends on the size of the angles of $ABCD$. Determine this ratio.

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60 Prove the inequality

$$(-a+b+c)^2(a-b+c)^2(a+b-c)^2 \geq (-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)$$

for all real numbers a, b, c.

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61 Given a positive integer n, find the greatest integer p with the property that for any function $f : \mathbb{P}(X) \to C$, where X and C are sets of cardinality n and p, respectively, there exist two distinct sets $A, B \in \mathbb{P}(X)$ such that $f(A) = f(B) = f(A \cup B)$. ($\mathbb{P}(X)$ is the family of all subsets of X.)

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62 Determine all pairs of positive integers (x, y) satisfying the equation $p^x - y^3 = 1$, where p is a given prime number.

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63 Let AA', BB', CC' be the bisectors of the angles of a triangle ABC ($A' \in BC, B' \in CA, C' \in AB$). Prove that each of the lines $A'B', B'C', C'A'$ intersects the incircle in two points.

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64 Let $(a_n)_{n\in\mathbb{N}}$ be the sequence of integers defined recursively by $a_1 = a_2 = 1, a_{n+2} = 7a_{n+1} - a_n - 2$ for $n \geq 1$. Prove that a_n is a perfect square for every n.

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65 Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a circle C. Show that there is a point M on C such that $MA_1 - MA_2 + MA_3 - MA_4 = 0$.

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66 One hundred red points and one hundred blue points are chosen in the plane, no three of them lying on a line. Show that these points can be connected pairwise, red ones with blue ones, by disjoint line segments.

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67 Let $f(x) = x^n$ where n is a fixed positive integer and $x = 1, 2, \cdots$. Is the decimal expansion $a = 0.f(1)f(2)f(3)...$ rational for any value of n ?

The decimal expansion of a is defined as follows: If $f(x) = d_1(x)d_2(x)\cdots d_{r(x)}(x)$ is the decimal expansion of $f(x)$, then $a = 0.1d_1(2)d_2(2)\cdots d_{r(2)}(2)d_1(3)...d_{r(3)}(3)d_1(4)\cdots$.

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68 Consider the equation $x^4 + ax^3 + bx^2 + ax + 1 = 0$ with real coefficients a, b. Determine the number of distinct real roots and their multiplicities for various values of a and b. Display your result graphically in the (a, b) plane.

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69 Let AX, BY, CZ be three cevians concurrent at an interior point D of a triangle ABC. Prove that if two of the quadrangles $DYAZ, DZBX, DXCY$ are circumscribable, so is the third.

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70 Let $ABCD$ be a tetrahedron having each sum of opposite sides equal to 1. Prove that

$$r_A + r_B + r_C + r_D \leq \frac{\sqrt{3}}{3}$$

where r_A, r_B, r_C, r_D are the inradii of the faces, equality holding only if $ABCD$ is regular.

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71 Two straight lines perpendicular to each other meet each side of a triangle in points symmetric with respect to the midpoint of that side. Prove that these two lines intersect in a point on the nine-point circle.

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72 A one-person game with two possible outcomes is played as follows:
After each play, the player receives either a or b points, where a and b are integers with $0 < b < a < 1986$. The game is played as many times as one wishes and the total score of the game is defined as the sum of points received after successive plays. It is observed that every integer $x \geq 1986$ can be obtained as the total score whereas 1985 and 663 cannot. Determine a and b.

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73 Let $(a_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers such that $\lim_{i\to\infty} a_i = +\infty$ and $a_{i+1}/a_i \leq 10$ for each i. Prove that for every positive integer k there are infinitely many pairs (i, j) with $10^k \leq a_i/a_j \leq 10^{k+1}$.

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74 From a collection of n persons q distinct two-member teams are selected and ranked $1, \cdots, q$ (no ties). Let m be the least integer larger than or equal to $2q/n$. Show that there are m distinct teams that may be listed so that :
(i) each pair of consecutive teams on the list have one member in common and
(ii) the chain of teams on the list are in rank order.

Alternative formulation.
Given a graph with n vertices and q edges numbered $1, \cdots, q$, show that there exists a chain of m edges, $m \geq \frac{2q}{n}$, each two consecutive edges having a common vertex, arranged monotonically with respect to the numbering.

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75 The incenter of a triangle is the midpoint of the line segment of length 4 joining the centroid and the orthocenter of the triangle. Determine the maximum possible area of the triangle.

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76 Let A, B, and C be three points on the edge of a circular chord such that B is due west of C and ABC is an equilateral triangle whose side is 86 meters long. A boy swam from A directly toward B. After covering a distance of x meters, he turned and swam westward, reaching the shore after covering a distance of y meters. If x and y are both positive integers, determine y.

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77 Find all integers x, y, z such that

$$x^3 + y^3 + z^3 = x + y + z = 8$$

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78 If T and T_1 are two triangles with angles x, y, z and x_1, y_1, z_1, respectively, prove the inequality

$$\frac{\cos x_1}{\sin x} + \frac{\cos y_1}{\sin y} + \frac{\cos z_1}{\sin z} \leq \cot x + \cot y + \cot z.$$

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79 Let AA_1, BB_1, CC_1 be the altitudes in an acute-angled triangle ABC, K and M are points on the line segments A_1C_1 and B_1C_1 respectively. Prove that if the angles MAK and CAA_1 are equal, then the angle C_1KM is bisected by AK.

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80 Let $ABCD$ be a tetrahedron and O its incenter, and let the line OD be perpendicular to AD. Find the angle between the planes DOB and DOC.
