IMO Longlists 1988

www.artofproblemsolving.com/community/c4019

by orl, Armo

1 An integer sequence is defined by

$$a_n = 2a_{n-1} + a_{n-2}, \quad (n > 1), \quad a_0 = 0, a_1 = 1.$$

Prove that 2^k divides a_n if and only if 2^k divides n.

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2 Let $\left[\sqrt{(n+1)^2+n^2}\right], n = 1, 2, \ldots,$ where $[x]$ denotes the integer part of x. Prove that

i.) there are infinitely many positive integers m such that $a_{m+1} - a_m > 1$;
ii.) there are infinitely many positive integers m such that $a_{m+1} - a_m = 1$.

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3 Let n be a positive integer. Find the number of odd coefficients of the polynomial

$$u_n(x) = (x^2 + x + 1)^n.$$

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4 The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B and C meet the circle again at A', B' and C' respectively. Prove that the area of triangle $A'B'C'$ is greater than or equal to the area of triangle ABC.

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5 Let k be a positive integer and M_k the set of all the integers that are between $2 \cdot k^2 + k$ and $2 \cdot k^2 + 3 \cdot k$, both included. Is it possible to partition M_k into 2 subsets A and B such that

$$\sum_{x \in A} x^2 = \sum_{x \in B} x^2.$$

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6 An $n \times n, n \geq 2$ chessboard is numbered by the numbers $1, 2, \ldots, n^2$ (and every number occurs). Prove that there exist two neighbouring (with common edge) squares such that their numbers differ by at least n.

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7 Let n be an even positive integer. Let $A_1, A_2, \ldots, A_{n+1}$ be sets having n elements each such that any two of them have exactly one element in common while every element of their union belongs to at least two of the given sets. For which n can one assign to every element of the union one of the numbers 0 and 1 in such a manner that each of the sets has exactly $\frac{n}{2}$ zeros?

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8 In a given tedrahedron $ABCD$ let K and L be the centres of edges AB and CD respectively. Prove that every plane that contains the line KL divides the tedrahedron into two parts of equal volume.

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9 If a_0 is a positive real number, consider the sequence $\{a_n\}$ defined by:

$$a_{n+1} = \frac{a_n^2 - 1}{n + 1}, n \geq 0.$$

Show that there exist a real number $a > 0$ such that:

i.) for all $a_0 \geq a,$ the sequence $\{a_n\} \rightarrow \infty,$

ii.) for all $a_0 < a,$ the sequence $\{a_n\} \rightarrow 0.$

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10 Let a be the greatest positive root of the equation $x^3 - 3 \cdot x^2 + 1 = 0.$ Show that $\left[a^{1788}\right]$ and $\left[a^{1988}\right]$ are both divisible by 17. Here $[x]$ denotes the integer part of $x.$

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11 Let $u_1, u_2, \ldots, u_m$ be m vectors in the plane, each of length $\leq 1,$ with zero sum. Show that one can arrange $u_1, u_2, \ldots, u_m$ as a sequence $v_1, v_2, \ldots, v_m$ such that each partial sum $v_1, v_1 + v_2, v_1 + v_2 + v_3, \ldots, v_1, v_2, \ldots, v_m$ has length less than or equal to $\sqrt{5}.$

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12 Show that there do not exist more than 27 half-lines (or rays) emanating from the origin in the 3-dimensional space, such that the angle between each pair of rays is $\geq \frac{\pi}{4}.$

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13 Let T be a triangle with inscribed circle $C.$ A square with sides of length a is circumscribed about the same circle $C.$ Show that the total length of the parts of the edge of the square interior to the triangle T is at least $2 \cdot a.$

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14 Let a and b be two positive integers such that $a \cdot b + 1$ divides $a^2 + b^2$. Show that $\frac{a^2+b^2}{a \cdot b+1}$ is a perfect square.

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15 Let $1 \leq k \leq n.$ Consider all finite sequences of positive integers with sum $n.$ Find $T(n,k),$ the total number of terms of size k in all of the sequences.

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16 If n runs through all the positive integers, then $f(n) = \left[n + \sqrt{\frac{n}{3}} + \frac{1}{2}\right]$ runs through all positive integers skipping the terms of the sequence $a_n = 3 \cdot n^2 - 2 \cdot n.$

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17 If n runs through all the positive integers, then $f(n) = \left\lfloor n + \sqrt{3n} + \frac{1}{2}\right\rfloor$ runs through all positive integers skipping the terms of the sequence $a_n = \left\lfloor \frac{n^2+2n}{3} \right\rfloor.$

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18 Let $N = \{1, 2 \ldots, n\}, n \geq 2$. A collection $F = \{A_1, \ldots, A_t\}$ of subsets $A_i \subseteq N$, $i = 1, \ldots, t$, is said to be separating, if for every pair $\{x, y\} \subseteq N$, there is a set $A_i \in F$ so that $A_i \cap \{x, y\}$ contains just one element. F is said to be covering, if every element of N is contained in at least one set $A_i \in F$. What is the smallest value $f(n)$ of t, so there is a set $F = \{A_1, \ldots, A_t\}$ which is simultaneously separating and covering?

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19 Let $Z_{m,n}$ be the set of all ordered pairs (i, j) with $i \in 1, \ldots, m$ and $j \in 1, \ldots, n$. Also let $a_{m,n}$ be the number of all those subsets of $Z_{m,n}$ that contain no 2 ordered pairs (i_1, j_1) and (i_2, j_2) with $|i_1 - i_2| + |j_1 - j_2| = 1$. Then show, for all positive integers m and k, that

$$a_{m,2\cdot k}^2 \leq a_{m,2\cdot k-1} \cdot a_{m,2\cdot k+1}.$$

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20 The lock of a safe consists of 3 wheels, each of which may be set in 8 different ways positions. Due to a defect in the safe mechanism the door will open if any two of the three wheels are in the correct position. What is the smallest number of combinations which must be tried if one is to guarantee being able to open the safe (assuming the "right combination" is not known)?

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21 Let "AB" and CD be two perpendicular chords of a circle with centre O and radius r and let X, Y, Z, W denote the cyclical order of the four parts into which the disc is thus divided. Find the maximum and minimum of the quantity

$$\frac{A(X) + A(Z)}{A(Y) + A(W)},$$

where $A(U)$ denotes the area of U.

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22 In a triangle ABC, choose any points $K \in BC, L \in AC, M \in AB, N \in LM, R \in MK$ and $F \in KL$. If $E_1, E_2, E_3, E_4, E_5, E_6$ and E denote the areas of the triangles $AMR, CKR, BKF, ALF, BNM, CLN$ and ABC respectively, show that

$$E \geq 8 \cdot \sqrt[6]{E_1 E_2 E_3 E_4 E_5 E_6}.$$

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23 In a right-angled triangle ABC let AD be the altitude drawn to the hypotenuse and let the straight line joining the incentres of the triangles ABD, ACD intersect the sides AB, AC at the points K, L respectively. If E and E_1 dnote the areas of triangles ABC and AKL respectively, show that

$$\frac{E}{E_1} \geq 2.$$

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24 Find the positive integers $x_1, x_2, \ldots, x_{29}$ at least one of which is greater that 1988 so that

$$x_1^2 + x_2^2 + \ldots x_{29}^2 = 29 \cdot x_1 \cdot x_2 \ldots x_{29}.$$

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25 Find the total number of different integers the function

$$f(x) = [x] + [2 \cdot x] + \left[\frac{5 \cdot x}{3}\right] + [3 \cdot x] + [4 \cdot x]$$

takes for $0 \leq x \leq 100.$

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26 The circle $x^2 + y^2 = r^2$ meets the coordinate axis at $A = (r, 0), B = (-r, 0), C = (0, r)$ and $D = (0, -r)$. Let $P = (u, v)$ and $Q = (-u, v)$ be two points on the circumference of the circle. Let N be the point of intersection of PQ and the y-axis, and M be the foot of the perpendicular drawn from P to the x-axis. If r^2 is odd, $u = p^m > q^n = v,$ where p and q are prime numbers and m and n are natural numbers, show that

$$|AM| = 1, |BM| = 9, |DN| = 8, |PQ| = 8.$$

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27 Assuming that the roots of $x^3 + p \cdot x^2 + q \cdot x + r = 0$ are real and positive, find a relation between p, q and r which gives a necessary condition for the roots to be exactly the cosines of the three angles of a triangle.

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28 Find a necessary and sufficient condition on the natural number n for the equation

$$x^n + (2 + x)^n + (2 - x)^n = 0$$

to have a integral root.

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29 Express the number 1988 as the sum of some positive integers in such a way that the product of these positive integers is maximal.

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30 In the triangle ABC let D, E and F be the mid-points of the three sides, X, Y and Z the feet of the three altitudes, H the orthocenter, and P, Q and R the mid-points of the line segment joining H to the three vertices. Show that the nine points $D, E, F, P, Q, R, X, Y, Z$ lie on a circle.

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31 For what values of n does there exist an $n \times n$ array of entries -1, 0 or 1 such that the $2 \cdot n$ sums obtained by summing the elements of the rows and the columns are all different?

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32 n points are given on the surface of a sphere. Show that the surface can be divided into n congruent regions such that each of them contains exactly one of the given points.

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33 In a multiple choice test there were 4 questions and 3 possible answers for each question. A group of students was tested and it turned out that for any three of them there was a question which the three students answered differently. What is the maximum number of students tested?

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34 Let ABC be an acute-angled triangle. The lines L_A, L_B and L_C are constructed through the vertices A, B and C respectively according the following prescription: Let H be the foot of the altitude drawn from the vertex A to the side BC; let S_A be the circle with diameter AH; let S_A meet the sides AB and AC at M and N respectively, where M and N are distinct from A; then let L_A be the line through A perpendicular to MN. The lines L_B and L_C are constructed similarly. Prove that the lines L_A, L_B and L_C are concurrent.

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35 A sequence of numbers $a_n, n = 1, 2, \ldots,$ is defined as follows: $a_1 = \frac{1}{2}$ and for each $n \geq 2$

$$a_n = \frac{2n-3}{2n} a_{n-1}.$$

Prove that $\sum_{k=1}^{n} a_k < 1$ for all $n \geq 1$.

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36 **i.)** Let ABC be a triangle with $AB = 12$ and $AC = 16$. Suppose M is the midpoint of side BC and points E and F are chosen on sides AC and AB, respectively, and suppose that lines EF and AM intersect at G. If $AE = 2 \cdot AF$ then find the ratio

$$\frac{EG}{GF}$$

ii.) Let E be a point external to a circle and suppose that two chords EAB and EDC meet at angle of 40°. If $AB = BC = CD$ find the size of angle ACD.

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37 **i.)** Four balls of radius 1 are mutually tangent, three resting on the floor and the fourth resting on the others. A tedrahedron, each of whose edges has length s, is circumscribed around the balls. Find the value of s.

ii.) Suppose that $ABCD$ and $EFGH$ are opposite faces of a retangular solid, with $\angle DHC = 45^{\circ}$ and $\angle FHB = 60^{\circ}$. Find the cosine of $\angle BHD$.

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38 **i.)** The polynomial $x^{2 \cdot k} + 1 + (x+1)^{2 \cdot k}$ is not divisible by $x^2 + x + 1$. Find the value of k.

ii.) If p, q and r are distinct roots of $x^3 - x^2 + x - 2 = 0$ the find the value of $p^3 + q^3 + r^3$.

iii.) If r is the remainder when each of the numbers 1059, 1417 and 2312 is divided by d, where

d is an integer greater than one, then find the value of $d - r$.

iv.) What is the smallest positive odd integer n such that the product of

$$2^{\frac{1}{7}}, 2^{\frac{3}{7}}, \ldots, 2^{\frac{2\cdot n+1}{7}}$$

is greater than 1000?

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39 **i.)** Let $g(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. What is the remainder when the polynomial $g(x^{12}$ is divided by the polynomial $g(x)$?

ii.) If k is a positive number and f is a function such that, for every positive number x, $f(x^2 + 1)^{\sqrt{x}} = k$. Find the value of

$$f(\frac{9+y^2}{y^2})^{\sqrt{\frac{12}{y}}}$$

for every positive number y.

iii.) The function f satisfies the functional equation $f(x) + f(y) = f(x + y) - x \cdot y - 1$ for every pair x, y of real numbers. If $f(1) = 1$, then find the numbers of integers n, for which $f(n) = n$.

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40 **i.)** Consider a circle K with diameter AB; with circle L tangent to AB and to K and with a circle M tangent to circle K, circle L and AB. Calculate the ration of the area of circle K to the area of circle M.

ii.) In triangle $ABC, AB = AC$ and $\angle CAB = 80^\circ$. If points D, E and F lie on sides BC, AC and AB, respectively and $CE = CD$ and $BF = BD$, then find the size of $\angle EDF$.

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41 **i.)** Calculate x if

$$x = \frac{(11 + 6\cdot\sqrt{2})\cdot\sqrt{11 - 6\cdot\sqrt{2}} - (11 - 6\cdot\sqrt{2})\cdot\sqrt{11 + 6\cdot\sqrt{2}}}{(\sqrt{\sqrt{5}+2} + \sqrt{\sqrt{5}-2}) - (\sqrt{\sqrt{5}+1})}$$

ii.) For each positive number x, let

$$k = \frac{\left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6}\right) - 2}{\left(x + \frac{1}{x}\right)^3 - \left(x^3 + \frac{1}{x^3}\right)}$$

Calculate the minimum value of k.

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42 Show that the solution set of the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose length is 1988.

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43 Find all plane triangles whose sides have integer length and whose incircles have unit radius.

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44 Let $-1 < x < 1$. Show that

$$\sum_{k=0}^{6} \frac{1-x^2}{1-2 \cdot x \cdot \cos\left(\frac{2 \cdot \pi \cdot k}{7}\right)+x^2} = \frac{7 \cdot \left(1+x^7\right)}{\left(1-x^7\right)}.$$

Deduce that

$$\csc^2\left(x+\frac{\pi}{7}\right)+\csc^2\left(2 \cdot x+\frac{\pi}{7}\right)+\csc^2\left(3 \cdot x+\frac{\pi}{7}\right) = 8.$$

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45 Let $g(n)$ be defined as follows:

$$g(1) = 0, g(2) = 1$$

and

$$g(n+2) = g(n) + g(n+1) + 1, n \geq 1.$$

Prove that if $n > 5$ is a prime, then n divides $g(n) \cdot (g(n)+1)$.

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46 $A_1, A_2, \ldots, A_{29}$ are 29 different sequences of positive integers. For $1 \leq i < j \leq 29$ and any natural number x, we define $N_i(x) =$ number of elements of the sequence A_i which are less or equal to x, and $N_{ij}(x) =$ number of elements of the intersection $A_i \cap A_j$ which are less than or equal to x. It is given for all $1 \leq i \leq 29$ and every natural number x,

$$N_i(x) \geq \frac{x}{e},$$

where $e = 2.71828\ldots$ Prove that there exist at least one pair i, j ($1 \leq i < j \leq 29$) such that

$$N_{ij}(1988) > 200.$$

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47 In the convex pentagon $ABCDE$, the sides BC, CD, DE are equal. Moreover each diagonal of the pentagon is parallel to a side (AC is parallel to DE, BD is parallel to AE etc.). Prove that $ABCDE$ is a regular pentagon.

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48 Consider 2 concentric circle radii R and r ($R > r$) with centre O. Fix P on the small circle and consider the variable chord PA of the small circle. Points B and C lie on the large circle; B, P, C are collinear and BC is perpendicular to AP.

i.) For which values of $\angle OPA$ is the sum $BC^2 + CA^2 + AB^2$ extremal?

ii.) What are the possible positions of the midpoints U of BA and V of AC as $\angle OPA$ varies?

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49 Let $f(n)$ be a function defined on the set of all positive integers and having its values in the same set. Suppose that $f(f(n)+f(m)) = m+n$ for all positive integers n, m. Find the possible value for $f(1988)$.

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50 Prove that the numbers A, B and C are equal, where:

- $A =$ number of ways that we can cover a $2 \times n$ rectangle with 2×1 retangles.
- $B =$ number of sequences of ones and twos that add up to n

- $C = \sum_{k=0}^{m} \binom{m+k}{2\cdot k}$ if $n = 2 \cdot m$, and
- $C = \sum_{k=0}^{m} \binom{m+k+1}{2\cdot k+1}$ if $n = 2 \cdot m + 1$.

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51 The positive integer n has the property that, in any set of n integers, chosen from the integers $1, 2, \ldots, 1988$, twenty-nine of them form an arithmetic progression. Prove that $n > 1788$.

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52 $ABCD$ is a quadrilateral. $A'BCD'$ is the reflection of $ABCD$ in BC, $A''B'CD'$ is the reflection of $A'BCD'$ in CD' and $A''B''C'D'$ is the reflection of $A''B'CD'$ in $D'A''$. Show that; if the lines AA'' and BB'' are parallel, then ABCD is a cyclic quadrilateral.

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53 Given n points $A_1, A_2, \ldots, A_n$, no three collinear, show that the n- gon $A_1A_2 \ldots A_n$, is inscribed in a circle if and only if
$A_1A_2 \cdot A_3A_n \cdot \ldots \cdot A_{n-1}A_n + A_2A_3 \cdot A_4A_n \cdot \ldots A_{n-1}A_n \cdot A_1A_n + \ldots + A_{n-1}A_{n-2} \cdot A_1A_n \cdot \ldots \cdot A_{n-3}A_n$
$= A_1A_{n-1} \cdot A_2A_n \cdot \ldots \cdot A_{n-2}A_n$,

where XY denotes the length of the segment XY.

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54 Find the least natural number n such that, if the set $\{1, 2, \ldots, n\}$ is arbitrarily divided into two non-intersecting subsets, then one of the subsets contains 3 distinct numbers such that the product of two of them equals the third.

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55 Suppose $\alpha_i > 0, \beta_i > 0$ for $1 \leq i \leq n, n > 1$ and that

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = \pi.$$

Prove that
$$\sum_{i=1}^{n} \frac{\cos(\beta_i)}{\sin(\alpha_i)} \leq \sum_{i=1}^{n} \cot(\alpha_i).$$

56 Given a set of 1988 points in the plane. No four points of the set are collinear. The points of a subset with 1788 points are coloured blue, the remaining 200 are coloured red. Prove that there exists a line in the plane such that each of the two parts into which the line divides the plane contains 894 blue points and 100 red points.

57 S is the set of all sequences $\{a_i | 1 \leq i \leq 7, a_i = 0$ or $1\}$. The distance between two elements $\{a_i\}$ and $\{b_i\}$ of S is defined as
$$\sum_{i=1}^{7} |a_i - b_i|.$$
T is a subset of S in which any two elements have a distance apart greater than or equal to 3. Prove that T contains at most 16 elements. Give an example of such a subset with 16 elements.

58 For a convex polygon P in the plane let P' denote the convex polygon with vertices at the mid-points of the sides of P. Given an integer $n \geq 3$, determine sharp bounds for the ratio
$$\frac{\text{area } P'}{\text{area } P},$$
over all convex n-gons P.

59 In 3-dimensional space there is given a point O and a finite set A of segments with the sum of lengths equal to 1988. Prove that there exists a plane disjoint from A such that the distance from it to O does not exceed 574.

60 Given integers $a_1, \ldots, a_{10}$, prove that there exist a non-zero sequence $\{x_1, \ldots, x_{10}\}$ such that all x_i belong to $\{-1, 0, 1\}$ and the number $\sum_{i=1}^{10} x_i \cdot a_i$ is divisible by 1001.

61 Forty-nine students solve a set of 3 problems. The score for each problem is a whole number of points from 0 to 7. Prove that there exist two students A and B such that, for each problem, A will score at least as many points as B.

62 Let $x = p, y = q, z = r, w = s$ be the unique solution of the system of linear equations
$$x + a_i \cdot y + a_i^2 \cdot z + a_i^3 \cdot w = a_i^4, i = 1, 2, 3, 4.$$
Express the solutions of the following system in terms of p, q, r and s :
$$x + a_i^2 \cdot y + a_i^4 \cdot z + a_i^6 \cdot w = a_i^8, i = 1, 2, 3, 4.$$

Assume the uniquness of the solution.

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63 Let p be the product of two consecutive integers greater than 2. Show that there are no integers $x_1, x_2, \ldots, x_p$ satisfying the equation

$$\sum_{i=1}^{p} x_i^2 - \frac{4}{4 \cdot p + 1} \left(\sum_{i=1}^{p} x_i \right)^2 = 1$$

OR

Show that there are only two values of p for which there are integers $x_1, x_2, \ldots, x_p$ satisfying

$$\sum_{i=1}^{p} x_i^2 - \frac{4}{4 \cdot p + 1} \left(\sum_{i=1}^{p} x_i \right)^2 = 1$$

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64 Find all positive integers x such that the product of all digits of x is given by $x^2 - 10 \cdot x - 22$.

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65 The Fibonacci sequence is defined by

$$a_{n+1} = a_n + a_{n-1}, n \geq 1, a_0 = 0, a_1 = a_2 = 1.$$

Find the greatest common divisor of the 1960-th and 1988-th terms of the Fibonacci sequence.

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66 Let C be a cube with edges of length 2. Construct a solid with fourteen faces by cutting off all eight corners at C, keeping the new faces perpendicular to the diagonals of the cube, and keeping the newly formed faces indentical. If at the conclusion of this process the fourteen faces so have the same area, find the area of each of face of the new solid.

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67 For each positive integer k and n, let $S_k(n)$ be the base k digit sum of n. Prove that there are at most two primes p less than $20,000$ for which $S_{31}(p)$ are composite numbers with at least two distinct prime divisors.

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68 In a group of n people, each one knows exactly three others. They are seated around a table. We say that the seating is $perfect$ if everyone knows the two sitting by their sides. Show that, if there is a perfect seating S for the group, then there is always another perfect seating which cannot be obtained from S by rotation or reflection.

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69 Let Q be the centre of the inscribed circle of a triangle ABC. Prove that for any point P,

$$a(PA)^2 + b(PB)^2 + c(PC)^2 = a(QA)^2 + b(QB)^2 + c(QC)^2 + (a+b+c)(QP)^2,$$

where $a = BC, b = CA$ and $c = AB$.

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70 ABC is a triangle, with inradius r and circumradius R. Show that:

$$\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)+\sin\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right)+\sin\left(\frac{C}{2}\right)\cdot\sin\left(\frac{A}{2}\right)\leq\frac{5}{8}+\frac{r}{4\cdot R}.$$

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71 The quadrilateral $A_1A_2A_3A_4$ is cyclic, and its sides are $a_1 = A_1A_2, a_2 = A_2A_3, a_3 = A_3A_4$ and $a_4 = A_4A_1$. The respective circles with centres I_i and radii r_i are tangent externally to each side a_i and to the sides a_{i+1} and a_{i-1} extended. ($a_0 = a_4$). Show that

$$\prod_{i=1}^{4}\frac{a_i}{r_i} = 4\cdot(\csc(A_1)+\csc(A_2))^2.$$

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72 Consider $h+1$ chess boards. Number the squares of each board from 1 to 64 in such a way that when the perimeters of any two boards of the collection are brought into coincidence in any possible manner, no two squares in the same position have the same number. What is the maximum value of h?

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73 A two-person game is played with nine boxes arranged in a 3×3 square and with white and black stones. At each move a player puts three stones, not necessarily of the same colour, in three boxes in either a horizontal or a vertical line. No box can contain stones of different colours: if, for instance, a player puts a white stone in a box containing black stones the white stone and one of the black stones are removed from the box. The game is over when the centrebox and the cornerboxes contain one black stone and the other boxes are empty. At one stage of a game x boxes contained one black stone each and the other boxes were empty. Determine all possible values for x.

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74 Let $\{a_k\}_1^\infty$ be a sequence of non-negative real numbers such that:

$$a_k - 2a_{k+1} + a_{k+2} \geq 0$$

and $\sum_{j=1}^{k} a_j \leq 1$ for all $k = 1, 2, \ldots$. Prove that:

$$0 \leq a_k - a_{k+1} < \frac{2}{k^2}$$

for all $k = 1, 2, \ldots$.

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75 Let S be an infinite set of integers containing zero, and such that the distances between successive number never exceed a given fixed number. Consider the following procedure: Given a set X of integers we construct a new set consisting of all numbers $x \pm s$, where x belongs to X and s belongs to S. Starting from $S_0 = \{0\}$ we successively construct sets $S_1, S_2, S_3, \ldots$ using this procedure. Show that after a finite number of steps we do not obtain any new sets, i.e. $S_k = S_{k_0}$ for $k \geq k_0$.

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76 A positive integer is called a **double number** if its decimal representation consists of a block of digits, not commencing with 0, followed immediately by an identical block. So, for instance, 360360 is a double number, but 36036 is not. Show that there are infinitely many double numbers which are perfect squares.

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77 A function f defined on the positive integers (and taking positive integers values) is given by:

$$f(1) = 1, f(3) = 3$$
$$f(2 \cdot n) = f(n)$$
$$f(4 \cdot n + 1) = 2 \cdot f(2 \cdot n + 1) - f(n)$$
$$f(4 \cdot n + 3) = 3 \cdot f(2 \cdot n + 1) - 2 \cdot f(n),$$

for all positive integers n. Determine with proof the number of positive integers ≤ 1988 for which $f(n) = n$.

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78 It is proposed to partition a set of positive integers into two disjoint subsets A and B subject to the conditions

i.) 1 is in A
ii.) no two distinct members of A have a sum of the form $2^k + 2, k = 0, 1, 2, \ldots;$ and
iii.) no two distinct members of B have a sum of that form.

Show that this partitioning can be carried out in unique manner and determine the subsets to which 1987, 1988 and 1989 belong.

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79 Let ABC be an acute-angled triangle. Let L be any line in the plane of the triangle ABC. Denote by u, v, w the lengths of the perpendiculars to L from A, B, C respectively. Prove the inequality $u^2 \cdot \tan A + v^2 \cdot \tan B + w^2 \cdot \tan C \geq 2 \cdot S$, where S is the area of the triangle ABC. Determine the lines L for which equality holds.

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80 The sequence $\{a_n\}$ of integers is defined by

$$a_1 = 2, a_2 = 7$$

and

$$-\frac{1}{2} < a_{n+1} - \frac{a_n^2}{a_{n-1}} \leq \ , n \geq 2.$$

Prove that a_n is odd for all $n > 1$.

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81 There are $n \geq 3$ job openings at a factory, ranked 1 to n in order of increasing pay. There are n job applicants, ranked from 1 to n in order of increasing ability. Applicant i is qualified for job j if and only if $i \geq j$. The applicants arrive one at a time in random order. Each in turn is hired to the highest-ranking job for which he or she is qualified AND which is lower in rank than any job already filled. (Under these rules, job 1 is always filled, and hiring terminates thereafter.) Show that applicants n and $n-1$ have the same probability of being hired.

82 The triangle ABC has a right angle at C. The point P is located on segment AC such that triangles PBA and PBC have congruent inscribed circles. Express the length $x = PC$ in terms of $a = BC, b = CA$ and $c = AB$.

83 A number of signal lights are equally spaced along a one-way railroad track, labeled in oder $1, 2, \dots, N, N \geq 2$. As a safety rule, a train is not allowed to pass a signal if any other train is in motion on the length of track between it and the following signal. However, there is no limit to the number of trains that can be parked motionless at a signal, one behind the other. (Assume the trains have zero length.) A series of K freight trains must be driven from Signal 1 to Signal N. Each train travels at a distinct but constant spped at all times when it is not blocked by the safety rule. Show that, regardless of the order in which the trains are arranged, the same time will elapse between the first train's departure from Signal 1 and the last train's arrival at Signal N.

84 A point M is chosen on the side AC of the triangle ABC in such a way that the radii of the circles inscribed in the triangles ABM and BMC are equal. Prove that

$$BM^2 = X \cot\left(\frac{B}{2}\right)$$

where X is the area of triangle ABC.

85 Around a circular table an even number of persons have a discussion. After a break they sit again around the circular table in a different order. Prove that there are at least two people such that the number of participants sitting between them before and after a break is the same.

86 Let a, b, c be integers different from zero. It is known that the equation $a \cdot x^2 + b \cdot y^2 + c \cdot z^2 = 0$ has a solution (x, y, z) in integer numbers different from the solutions $x = y = z = 0$. Prove that the equation

$$a \cdot x^2 + b \cdot y^2 + c \cdot z^2 = 1$$

has a solution in rational numbers.

87 In a row written in increasing order all the irreducible positive rational numbers, such that the product of the numerator and the denominator is less than 1988. Prove that any two adjacent fractions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} < \frac{c}{d}$, satisfy the equation $b \cdot c - a \cdot d = 1$.

88 Seven circles are given. That is, there are six circles inside a fixed circle, each tangent to the fixed circle and tangent to the two other adjacent smaller circles. If the points of contact between the six circles and the larger circle are, in order, A_1, A_2, A_3, A_4, A_5 and A_6 prove that

$$A_1A_2 \cdot A_3A_4 \cdot A_5A_6 = A_2A_3 \cdot A_4A_5 \cdot A_6A_1.$$

89 We match sets M of points in the coordinate plane to sets $M*$ according to the rule that $(x*, y*) \in M*$ if and only if $x \cdot x * + y \cdot y* \leq 1$ whenever $(x, y) \in M$. Find all triangles Q such that $Q*$ is the reflection of Q in the origin.

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90 Does there exist a number $\alpha, 0 < \alpha < 1$ such that there is an infinite sequence $\{a_n\}$ of positive numbers satisfying

$$1 + a_{n+1} \leq a_n + \frac{\alpha}{n} \cdot \alpha_n, n = 1, 2, \ldots ?$$

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91 A regular 14-gon with side a is inscribed in a circle of radius one. Prove

$$\frac{2-a}{2 \cdot a} > \sqrt{3 \cdot \cos\left(\frac{\pi}{7}\right)}.$$

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92 Let $p \geq 2$ be a natural number. Prove that there exist an integer n_0 such that

$$\sum_{i=1}^{n_0} \frac{1}{i \cdot \sqrt[p]{i+1}} > p.$$

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93 Given a natural number n, find all polynomials $P(x)$ of degree less than n satisfying the following condition

$$\sum_{i=0}^{n} P(i) \cdot (-1)^i \cdot \binom{n}{i} = 0.$$

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94 Let $n+1, n \geq 1$ positive integers be formed by taking the product of n given prime numbers (a prime number can appear several times or also not appear at all in a product formed in this way.) Prove that among these $n+1$ one can find some numbers whose product is a perfect square.
