IMO Longlists 1992

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1 Points D and E are chosen on the sides AB and AC of the triangle ABC in such a way that if F is the intersection point of BE and CD, then $AE+EF=AD+DF$. Prove that $AC+CF=AB+BF$.

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2 Let m be a positive integer and x_0, y_0 integers such that x_0, y_0 are relatively prime, y_0 divides x_0^2+m, and x_0 divides y_0^2+m. Prove that there exist positive integers x and y such that x and y are relatively prime, y divides x^2+m, x divides y^2+m, and $x+y\leq m+1$.

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3 Let ABC be a triangle, O its circumcenter, S its centroid, and H its orthocenter. Denote by A_1, B_1, and C_1 the centers of the circles circumscribed about the triangles CHB, CHA, and AHB, respectively. Prove that the triangle ABC is congruent to the triangle $A_1B_1C_1$ and that the nine-point circle of $\triangle ABC$ is also the nine-point circle of $\triangle A_1B_1C_1$.

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4 Let p,q, and r be the angles of a triangle, and let $a=\sin 2p, b=\sin 2q$, and $c=\sin 2r$. If $s=\frac{(a+b+c)}{2}$, show that

$$s(s-a)(s-b)(s-c)\geq 0.$$

When does equality hold?

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5 Let I,H,O be the incenter, centroid, and circumcenter of the nonisosceles triangle ABC. Prove that $AI\parallel HO$ if and only if $\angle BAC=120^\circ$.

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6 Suppose that n numbers $x_1,x_2,...,x_n$ are chosen randomly from the set $\{1,2,3,4,5\}$. Prove that the probability that $x_1^2+x_2^2+\cdots+x_n^2\equiv 0 \pmod 5$ is at least $\frac{1}{5}$.

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7 Let X be a bounded, nonempty set of points in the Cartesian plane. Let $f(X)$ be the set of all points that are at a distance of at most 1 from some point in X. Let $f_n(X)=f(f(\cdots(f(X))\cdots))$ (n times). Show that $f_n(X)$ becomes more circular as n gets larger.
In other words, if $r_n=\sup\{$radii of circles contained in $f_n(X)\}$ and $R_n=\inf\{$radii of circles containing $f_n($
then show that R_n/r_n gets arbitrarily close to 1 as n becomes arbitrarily large.

I'm not sure that I'm posting this in a right forum. If it's in a wrong forum, please mods move it.

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8 Given two positive real numbers a and b, suppose that a mapping $f:\mathbb{R}^+\to\mathbb{R}^+$ satisfies the functional equation

$$f(f(x))+af(x)=b(a+b)x.$$

Prove that there exists a unique solution of this equation.

9 The diagonals of a quadrilateral $ABCD$ are perpendicular: $AC \perp BD$. Four squares, $ABEF, BCGH, CDIJ$ are erected externally on its sides. The intersection points of the pairs of straight lines $CL, DF; DF, AH; A$ are denoted by P_1, Q_1, R_1, S_1, respectively, and the intersection points of the pairs of straight lines $AI, BK; BK, CE; CE, DG; DG, AI$ are denoted by P_2, Q_2, R_2, S_2, respectively. Prove that $P_1Q_1R_1S_1 \cong P_2Q_2R_2S_2$.

10 Consider 9 points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

11 Let $\phi(n, m), m \neq 1$, be the number of positive integers less than or equal to n that are coprime with m. Clearly, $\phi(m, m) = \phi(m)$, where $\phi(m)$ is Eulers phi function. Find all integers m that satisfy the following inequality:

$$\frac{\phi(n, m)}{n} \geq \frac{\phi(m)}{m}$$

for every positive integer n.

12 Given a triangle ABC such that the circumcenter is in the interior of the incircle, prove that the triangle ABC is acute-angled.

13 Let $ABCD$ be a convex quadrilateral such that $AC = BD$. Equilateral triangles are constructed on the sides of the quadrilateral. Let O_1, O_2, O_3, O_4 be the centers of the triangles constructed on AB, BC, CD, DA respectively. Show that O_1O_3 is perpendicular to O_2O_4.

14 Integers $a_1, a_2, ..., a_n$ satisfy $|a_k| = 1$ and

$$\sum_{k=1}^{n} a_k a_{k+1} a_{k+2} a_{k+3} = 2,$$

where $a_{n+j} = a_j$. Prove that $n \neq 1992$.

15 Prove that there exist 78 lines in the plane such that they have exactly 1992 points of intersection.

16 Find all triples (x, y, z) of integers such that

$$\frac{1}{x^2} + \frac{2}{y^2} + \frac{3}{z^2} = \frac{2}{3}$$

17 In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

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18 Fibonacci numbers are defined as follows: $F_0 = F_1 = 1, F_{n+2} = F_{n+1} + F_n, n \geq 0$. Let a_n be the number of words that consist of n letters 0 or 1 and contain no two letters 1 at distance two from each other. Express a_n in terms of Fibonacci numbers.

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19 Denote by a_n the greatest number that is not divisible by 3 and that divides n. Consider the sequence $s_0 = 0, s_n = a_1 + a_2 + \cdots + a_n, n \in \mathbb{N}$. Denote by $A(n)$ the number of all sums s_k $(0 \leq k \leq 3^n, k \in \mathbb{N}_0)$ that are divisible by 3. Prove the formula

$$A(n) = 3^{n-1} + 2 \cdot 3^{(n/2)-1} \cos\left(\frac{n\pi}{6}\right), \qquad n \in \mathbb{N}_0.$$

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20 Let X and Y be two sets of points in the plane and M be a set of segments connecting points from X and Y . Let k be a natural number. Prove that the segments from M can be painted using k colors in such a way that for any point $x \in X \cup Y$ and two colors α and β $(\alpha \neq \beta)$, the difference between the number of α-colored segments and the number of β-colored segments originating in X is less than or equal to 1.

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21 Prove that if $x, y, z > 1$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$, then

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

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22 For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.

a.) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.
b.) Find an integer n such that $S(n) = n^2 - 14$.
c.) Prove that there are infintely many integers n such that $S(n) = n^2 - 14$.

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23 An *Egyptian number* is a positive integer that can be expressed as a sum of positive integers, not necessarily distinct, such that the sum of their reciprocals is 1. For example, $32 = 2 + 3 + 9 + 18$ is Egyptian because $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18} = 1$. Prove that all integers greater than 23 are *Egyptian*.

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24 *(a)* Show that there exists exactly one function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying the following conditions:
(i) if $0 < q < \frac{1}{2}$, then $f(q) = 1 + f\left(\frac{q}{1-2q}\right)$;

(ii) if $1 < q \leq 2$, then $f(q) = 1 + f(q+1)$;

(iii) $f(q)f(1/q) = 1$ for all $q \in \mathbb{Q}^+$.

(b) Find the smallest rational number $q \in \mathbb{Q}^+$ such that $f(q) = \frac{19}{92}$.

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25 ***(a)*** Show that the set $\mathbb{N}$ of all positive integers can be partitioned into three disjoint subsets A, B, and C satisfying the following conditions:

$$A^2 = A, B^2 = C, C^2 = B,$$

$$AB = B, AC = C, BC = A,$$

where HK stands for $\{hk|h \in H, k \in K\}$ for any two subsets H, K of $\mathbb{N}$, and H^2 denotes HH.
(b) Show that for every such partition of $\mathbb{N}$, $\min\{n \in N | n \in A$ and $n + 1 \in A\}$ is less than or equal to 77.

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26 Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^2 + f(y)\right) = y + (f(x))^2 \quad \text{for all } x, y \in \mathbb{R}.$$

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27 Let ABC be an arbitrary scalene triangle. Define $\sum$ to be the set of all circles y that have the following properties:

(i) y meets each side of ABC in two (possibly coincident) points;

(ii) if the points of intersection of y with the sides of the triangle are labeled by P, Q, R, S, T, U, with the points occurring on the sides in orders $\mathcal{B}(B, P, Q, C), \mathcal{B}(C, R, S, A), \mathcal{B}(A, T, U, B)$, then the following relations of parallelism hold: $TS \parallel BC; PU \parallel CA; RQ \parallel AB$. (In the limiting cases, some of the conditions of parallelism will hold vacuously; e.g., if A lies on the circle y, then T , S both coincide with A and the relation $TS \parallel BC$ holds vacuously.)

(a) Under what circumstances is $\sum$ nonempty?

(b) Assuming that is nonempty, show how to construct the locus of centers of the circles in the set $\sum$.

(c) Given that the set $\sum$has just one element, deduce the size of the largest angle of ABC.
(d) Show how to construct the circles in $\sum$ that have, respectively, the largest and the smallest radii.

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28 Two circles Ω_1 and Ω_2 are externally tangent to each other at a point I, and both of these circles are tangent to a third circle Ω which encloses the two circles Ω_1 and Ω_2.
The common tangent to the two circles Ω_1 and Ω_2 at the point I meets the circle Ω at a point A.
One common tangent to the circles Ω_1 and Ω_2 which doesn't pass through I meets the circle Ω at the points B and C such that the points A and I lie on the same side of the line BC.

Prove that the point I is the incenter of triangle ABC.

Alternative formulation. Two circles touch externally at a point I. The two circles lie inside a large circle and both touch it. The chord BC of the large circle touches both smaller circles (not at I). The common tangent to the two smaller circles at the point I meets the large circle at a point A, where the points A and I are on the same side of the chord BC. Show that the point I is the incenter of triangle ABC.

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29 Show that in the plane there exists a convex polygon of 1992 sides satisfying the following conditions:

(i) its side lengths are $1, 2, 3, \ldots, 1992$ in some order;
(ii) the polygon is circumscribable about a circle.

Alternative formulation: Does there exist a 1992-gon with side lengths $1, 2, 3, \ldots, 1992$ circumscribed about a circle? Answer the same question for a 1990-gon.

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30 Let $P_n = (19 + 92)(19^2 + 92^2) \cdots (19^n + 92^n)$ for each positive integer n. Determine, with proof, the least positive integer m, if it exists, for which P_m is divisible by 33^{33}.

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31 Let $f(x) = x^8 + 4x^6 + 2x^4 + 28x^2 + 1$. Let $p > 3$ be a prime and suppose there exists an integer z such that p divides $f(z)$. Prove that there exist integers $z_1, z_2, \ldots, z_8$ such that if

$$g(x) = (x - z_1)(x - z_2) \cdot \ldots \cdot (x - z_8),$$

then all coefficients of $f(x) - g(x)$ are divisible by p.

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32 Let $S_n = \{1, 2, \cdots, n\}$ and $f_n : S_n \to S_n$ be defined inductively as follows: $f_1(1) = 1, f_n(2j) = j \ (j = 1, 2, \cdots, [n/2])$ and

-***(i)*** if $n = 2k \ (k \geq 1)$, then $f_n(2j - 1) = f_k(j) + k \ (j = 1, 2, \cdots, k)$;

-***(ii)*** if $n = 2k + 1 \ (k \geq 1)$, then $f_n(2k + 1) = k + f_{k+1}(1), f_n(2j - 1) = k + f_{k+1}(j + 1) \ (j = 1, 2, \cdots, k)$.

Prove that $f_n(x) = x$ if and only if x is an integer of the form

$$\frac{(2n + 1)(2^d - 1)}{2^{d+1} - 1}$$

for some positive integer d.

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33 Let a, b, c be positive real numbers and p, q, r complex numbers. Let S be the set of all solutions (x, y, z) in $\mathbb{C}$ of the system of simultaneous equations

$$ax + by + cz = p,$$

$$ax2 + by2 + cz2 = q,$$

$$ax3 + bx3 + cx3 = r.$$

Prove that S has at most six elements.

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34 Let a, b, c be integers. Prove that there are integers $p_1, q_1, r_1, p_2, q_2, r_2$ such that

$$a = q_1r_2 - q_2r_1, b = r_1p_2 - r_2p_1, c = p_1q_2 - p_2q_1.$$

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35 Let $f(x)$ be a polynomial with rational coefficients and α be a real number such that

$$\alpha^3 - \alpha = [f(\alpha)]^3 - f(\alpha) = 33^{1992}.$$

Prove that for each $n \geq 1$,

$$[f^n(\alpha)]^3 - f^n(\alpha) = 33^{1992},$$

where $f^n(x) = f(f(\cdots f(x)))$, and n is a positive integer.

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36 Find all rational solutions of

$$a^2 + c^2 + 17(b^2 + d^2) = 21,$$

$$ab + cd = 2.$$

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37 Let the circles C_1, C_2, and C_3 be orthogonal to the circle C and intersect each other inside C forming acute angles of measures A, B, and C. Show that $A + B + C < \pi$.

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38 Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set A.

Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.

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39 Let $n \geq 2$ be an integer. Find the minimum k for which there exists a partition of $\{1, 2, ..., k\}$ into n subsets $X_1, X_2, \cdots, X_n$ such that the following condition holds:
for any $i, j, 1 \leq i < j \leq n$, there exist $x_i \in X_1, x_j \in X_2$ such that $|x_i - x_j| = 1$.

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40 The colonizers of a spherical planet have decided to build N towns, each having area $1/1000$ of the total area of the planet. They also decided that any two points belonging to different towns will have different latitude and different longitude. What is the maximal value of N?

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41 Let S be a set of positive integers $n_1, n_2, \cdots, n_6$ and let $n(f)$ denote the number $n_1 n_{f(1)} + n_2 n_{f(2)} + \cdots + n_6 n_{f(6)}$, where f is a permutation of $\{1, 2, ..., 6\}$. Let

$$\Omega = \{n(f) | f \text{ is a permutation of } \{1, 2, ..., 6\}\}$$

Give an example of positive integers $n_1, \cdots, n_6$ such that Ω contains as many elements as possible and determine the number of elements of Ω.

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42 In a triangle ABC, let D and E be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with the sides AC, AB, respectively. Determine the angles $\angle A, \angle B, \angle C$ if $\angle BDE = 24^\circ$, $\angle CED = 18^\circ$.

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43 Find the number of positive integers n satisfying $\phi(n) | n$ such that

$$\sum_{m=1}^{\infty} \left(\left[\frac{n}{m} \right] - \left[\frac{n-1}{m} \right] \right) = 1992$$

What is the largest number among them? As usual, $\phi(n)$ is the number of positive integers less than or equal to n and relatively prime to n.

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44 Prove that $\frac{5^{125}-1}{5^{25}-1}$ is a composite number.

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45 Let n be a positive integer. Prove that the number of ways to express n as a sum of distinct positive integers (up to order) and the number of ways to express n as a sum of odd positive integers (up to order) are the same.

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46 Prove that the sequence $5, 12, 19, 26, 33, \cdots$ contains no term of the form $2^n - 1$.

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47 Evaluate

$$\left\lfloor \prod_{n=1}^{1992} \frac{3n+2}{3n+1} \right\rfloor$$

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48 Find all the functions $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying the identity

$$f(x)f(y) = y^\alpha f\left(\frac{x}{2}\right) + x^\beta f\left(\frac{y}{2}\right) \qquad \forall x, y \in \mathbb{R}^+$$

Where α, β are given real numbers.

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49 Given real numbers x_i $(i = 1, 2, \cdots, 4k+2)$ such that

$$\sum_{i=1}^{4k+2}(-1)^{i+1}x_i x_{i+1} = 4m \qquad (\ x_1 = x_{4k+3}\ )$$

prove that it is possible to choose numbers $x_{k_1}, \cdots, x_{k_6}$ such that

$$\sum_{i=1}^{6}(-1)^{i}k_i k_{i+1} > m \qquad (\ x_{k_1} = x_{k_7}\ )$$

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50 Let N be a point inside the triangle ABC. Through the midpoints of the segments AN, BN, and CN the lines parallel to the opposite sides of $\triangle ABC$ are constructed. Let AN, BN, and CN be the intersection points of these lines. If N is the orthocenter of the triangle ABC, prove that the nine-point circles of $\triangle ABC$ and $\triangle A_N B_N C_N$ coincide.

Remark. The statement of the original problem was that the nine-point circles of the triangles $A_N B_N C_N$ and $A_M B_M C_M$ coincide, where N and M are the orthocenter and the centroid of ABC. This statement is false.

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51 Let f, g and a be polynomials with real coefficients, f and g in one variable and a in two variables. Suppose

$$f(x) - f(y) = a(x, y)(g(x) - g(y)) \forall x, y \in \mathbb{R}$$

Prove that there exists a polynomial h with $f(x) = h(g(x))\ \forall x \in \mathbb{R}$.

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52 Let n be an integer > 1. In a circular arrangement of n lamps $L_0, \cdots, L_{n-1}$, each one of which can be either ON or OFF, we start with the situation that all lamps are ON, and then carry out a sequence of steps, $Step_0, Step_1, \cdots$. If L_{j-1} (j is taken mod n) is ON, then $Step_j$ changes the status of L_j (it goes from ON to OFF or from OFF to ON) but does not change the status of any of the other lamps. If L_{j-1} is OFF, then $Step_j$ does not change anything at all. Show that:

(a) There is a positive integer $M(n)$ such that after $M(n)$ steps all lamps are ON again.

(b) If n has the form 2^k, then all lamps are ON after $n^2 - 1$ steps.

(c) If n has the form $2^k + 1$, then all lamps are ON after $n^2 - n + 1$ steps.

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53 Find all integers a, b, c with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1)$$

is a divisor of $abc - 1$.

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54 Suppose that $n > m \geq 1$ are integers such that the string of digits 143 occurs somewhere in the decimal representation of the fraction $\frac{m}{n}$. Prove that $n > 125.$

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55 For any positive integer x define $g(x)$ as greatest odd divisor of x, and

$$f(x) = \begin{cases} \frac{x}{2} + \frac{x}{g(x)} & \text{if } x \text{ is even,} \\ 2^{\frac{x+1}{2}} & \text{if } x \text{ is odd.} \end{cases}$$

Construct the sequence $x_1 = 1, x_{n+1} = f(x_n)$. Show that the number 1992 appears in this sequence, determine the least n such that $x_n = 1992,$ and determine whether n is unique.

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56 A directed graph (any two distinct vertices joined by at most one directed line) has the following property: If $x, u,$ and v are three distinct vertices such that $x \to u$ and $x \to v$, then $u \to w$ and $v \to w$ for some vertex $w.$ Suppose that $x \to u \to y \to \cdots \to z$ is a path of length n, that cannot be extended to the right (no arrow goes away from z). Prove that every path beginning at x arrives after n steps at $z.$

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57 For positive numbers a, b, c define $A = \frac{(a+b+c)}{3}$, $G = \sqrt[3]{abc}$, $H = \frac{3}{(a^{-1}+b^{-1}+c^{-1})}$. Prove that

$$\left(\frac{A}{G}\right)^3 \geq \frac{1}{4} + \frac{3}{4} \cdot \frac{A}{H}.$$

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59 Let a regular 7-gon $A_0A_1A_2A_3A_4A_5A_6$ be inscribed in a circle. Prove that for any two points P, Q on the arc A_0A_6 the following equality holds:

$$\sum_{i=0}^{6} (-1)^i PA_i = \sum_{i=0}^{6} (-1)^i QA_i.$$

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60 Does there exist a set M with the following properties?

(i) The set M consists of 1992 natural numbers.
(ii) Every element in M and the sum of any number of elements have the form m^k $(m, k \in \mathbb{N}, k \geq 2).$

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61 There are a board with $2n \cdot 2n$ $(= 4n^2)$ squares and $4n^2 - 1$ cards numbered with different natural numbers. These cards are put one by one on each of the squares. One square is empty. We can move a card to an empty square from one of the adjacent squares (two squares are adjacent if they have a common edge). Is it possible to exchange two cards on two adjacent squares of a column (or a row) in a finite number of movements?

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62 Let $c_1, \cdots, c_n$ $(n \geq 2)$ be real numbers such that $0 \leq \sum c_i \leq n$. Prove that there exist integers $k_1, \cdots, k_n$ such that $\sum k_i = 0$ and $1 - n \leq c_i + nk_i \leq n$ for every $i = 1, \cdots, n$.

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63 Let a and b be integers. Prove that $\frac{2a^2-1}{b^2+2}$ is not an integer.

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64 For any positive integer n consider all representations $n = a_1 + \cdots + a_k$, where $a_1 > a_2 > \cdots > a_k > 0$ are integers such that for all $i \in \{1, 2, \cdots, k-1\}$, the number a_i is divisible by a_{i+1}. Find the longest such representation of the number 1992.

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65 If A, B, C, and D are four distinct points in space, prove that there is a plane P on which the orthogonal projections of A, B, C, and D form a parallelogram (possibly degenerate).

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66 A circle of radius ρ is tangent to the sides AB and AC of the triangle ABC, and its center K is at a distance p from BC.

(a) Prove that $a(p - \rho) = 2s(r - \rho)$, where r is the inradius and $2s$ the perimeter of ABC.
(b) Prove that if the circle intersect BC at D and E, then

$$DE = \frac{4\sqrt{rr_1(\rho - r)(r_1 - \rho)}}{r_1 - r}$$

where r_1 is the exradius corresponding to the vertex A.

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67 In a triangle, a symmedian is a line through a vertex that is symmetric to the median with the respect to the internal bisector (all relative to the same vertex). In the triangle ABC, the median m_a meets BC at A' and the circumcircle again at A_1. The symmedian s_a meets BC at M and the circumcircle again at A_2. Given that the line A_1A_2 contains the circumcenter O of the triangle, prove that:

(a) $\frac{AA'}{AM} = \frac{b^2+c^2}{2bc}$;

(b) $1 + 4b^2c^2 = a^2(b^2 + c^2)$

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68 Show that the numbers $\tan\left(\frac{r\pi}{15}\right)$, where r is a positive integer less than 15 and relatively prime to 15, satisfy

$$x^8 - 92x^6 + 134x^4 - 28x^2 + 1 = 0.$$

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69 Let $\alpha(n)$ be the number of digits equal to one in the binary representation of a positive integer n. Prove that:

(a) the inequality $\alpha(n)(n^2) \leq \frac{1}{2}\alpha(n)(\alpha(n) + 1)$ holds;
(b) the above inequality is an equality for infinitely many positive integers, and

(c) there exists a sequence $(n_i)_1^\infty$ such that $\frac{\alpha(n_i^2)}{\alpha(n_i}$
goes to zero as i goes to ∞.

Alternative problem: Prove that there exists a sequence a sequence $(n_i)_1^\infty$ such that $\frac{\alpha(n_i^2)}{\alpha(n_i)}$

(d) ∞;
(e) an arbitrary real number $\gamma \in (0,1)$;
(f) an arbitrary real number $\gamma \geq 0$;

as i goes to ∞.

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70 Let two circles A and B with unequal radii r and R, respectively, be tangent internally at the point A_0. If there exists a sequence of distinct circles (C_n) such that each circle is tangent to both A and B, and each circle C_{n+1} touches circle C_n at the point A_n, prove that

$$\sum_{n=1}^{\infty} |A_{n+1}A_n| < \frac{4\pi Rr}{R+r}.$$

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71 Let $P_1(x,y)$ and $P_2(x,y)$ be two relatively prime polynomials with complex coefficients. Let $Q(x,y)$ and $R(x,y)$ be polynomials with complex coefficients and each of degree not exceeding d. Prove that there exist two integers A_1, A_2 not simultaneously zero with $|A_i| \leq d+1$ $(i = 1,2)$ and such that the polynomial $A_1P_1(x,y) + A_2P_2(x,y)$ is coprime to $Q(x,y)$ and $R(x,y)$.

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72 In a school six different courses are taught: mathematics, physics, biology, music, history, geography. The students were required to rank these courses according to their preferences, where equal preferences were allowed. It turned out that:

-**(i)** mathematics was ranked among the most preferred courses by all students;

-**(ii)** no student ranked music among the least preferred ones;

-**(iii)** all students preferred history to geography and physics to biology; and

-**(iv)** no two rankings were the same.

Find the greatest possible value for the number of students in this school.

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73 Let $\{A_n | n = 1, 2, \cdots\}$ be a set of points in the plane such that for each n, the disk with center A_n and radius 2^n contains no other point A_j . For any given positive real numbers $a < b$ and

R, show that there is a subset G of the plane satisfying:

(i) the area of G is greater than or equal to R;

(ii) for each point P in G, $a < \sum_{n=1}^{\infty} \frac{1}{|A_nP|} < b$.

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74 Let $S = \{\frac{\pi^n}{1992^m} | m, n \in \mathbb{Z}\}$. Show that every real number $x \geq 0$ is an accumulation point of S.

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75 A sequence $\{an\}$ of positive integers is defined by

$$a_n = \left[n + \sqrt{n} + \frac{1}{2} \right], \qquad \forall n \in \mathbb{N}$$

Determine the positive integers that occur in the sequence.

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76 Given any triangle ABC and any positive integer n, we say that n is a *decomposable* number for triangle ABC if there exists a decomposition of the triangle ABC into n subtriangles with each subtriangle similar to $\triangle ABC$. Determine the positive integers that are decomposable numbers for every triangle.

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77 Show that if 994 integers are chosen from $1, 2, \cdots, 1992$ and one of the chosen integers is less than 64, then there exist two among the chosen integers such that one of them is a factor of the other.

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78 Let F_n be the nth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. Let $A_0, A_1, A_2, \cdots$ be a sequence of points on a circle of radius 1 such that the minor arc from A_{k-1} to A_k runs clockwise and such that

$$\mu(A_{k-1}A_k) = \frac{4F_{2k+1}}{F_{2k+1}^2 + 1}$$

for $k \geq 1$, where $\mu(XY)$ denotes the radian measure of the arc XY in the clockwise direction. What is the limit of the radian measure of arc A_0A_n as n approaches infinity?

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79 Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Pick any x_1 in $[0, 1)$ and define the sequence $x_1, x_2, x_3, \ldots$ by $x_{n+1} = 0$ if $x_n = 0$ and $x_{n+1} = \frac{1}{x_n} - \left\lfloor \frac{1}{x_n} \right\rfloor$ otherwise. Prove that

$$x_1 + x_2 + \ldots + x_n < \frac{F_1}{F_2} + \frac{F_2}{F_3} + \ldots + \frac{F_n}{F_{n+1}},$$

where $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

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80 Given a graph with n vertices and a positive integer m that is less than n, prove that the graph contains a set of $m + 1$ vertices in which the difference between the largest degree of any vertex in the set and the smallest degree of any vertex in the set is at most $m - 1$.

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81 Suppose that points X, Y, Z are located on sides BC, CA, and AB, respectively, of triangle ABC in such a way that triangle XYZ is similar to triangle ABC. Prove that the orthocenter of triangle XYZ is the circumcenter of triangle ABC.

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82 Let $f(x) = x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$ and $g(x) = x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ be two polynomials with real coefficients such that for each real number x, $f(x)$ is the square of an integer if and only if so is $g(x)$. Prove that if $n + m > 0$, then there exists a polynomial $h(x)$ with real coefficients such that $f(x) \cdot g(x) = (h(x))^2$.

Remark. The original problem stated $g(x) = x^n + b_1x^{n-1} + \cdots + b_{n-1} + b_n$, but I think the right form of the problem is what I wrote.
