

Today's Calculation Of Integral 2008

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257 Evaluate $\sqrt{\pi \int_0^{2008} x |\sin \pi x| dx}$

258 Find the volume of the solid formed by the revolution of the curve $x = a(1 + \cos \theta) \cos \theta$, $y = a(1 + \cos \theta) \sin \theta$ ($a > 0$, $-\pi < \theta \leq \pi$) about the x axis.

259 Evaluate $\int_0^{\frac{\pi}{12}} \frac{dx}{(\sin x + \cos x)^4}$

260 Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{(\sin^3 \theta - \cos^3 \theta - \cos^2 \theta)(\sin \theta + \cos \theta + \cos^2 \theta)^{2007}}{\sin^{2009} \theta \cos^{2009} \theta} d\theta$

261 Find the continuous function $f(x)$ such that $f(x) = \int_0^x \{f(t) \cos t - \cos(t - x)\} dt$.

262 Answer the following questions for positive integer n .

(1) Find the maximum value of $f_n(x) = x^n e^{-x}$ for $x \geq 0$.

(2) Show that $\lim_{x \rightarrow \infty} f_n(x) = 0$.

(3) Let $I_n(x) = \int_0^x f_n(t) dt$, find $\lim_{x \rightarrow \infty} I_n(x)$.

263 Let $F(t) = \frac{1}{t} \int_0^{\frac{\pi}{2}t} |\cos 2x| dx$ for $0 < t \leq 1$.

(1) Find $\lim_{t \rightarrow 0} F(t)$.

(2) Find the range of t such that $F(t) \geq 1$.

264 Find the area of the figure surrounded by the locus of the point P inside the square with side length a such that the distance to the center of the square is equal to the minimum distance to the side of the square.

265 Supposed that $f(x)$ has $f'(x)$ and for any real numbers x, y , $\int_y^{x+y} f(t) dt = \int_0^x \{f(y) \cos t + f(t) \cos y\} dt$ holds.

(1) Express $f(x + y)$ in terms of $f(x)$, $f(y)$.

(2) Find $f'(x + \frac{\pi}{2})$. Note that $f(\frac{\pi}{2}) = 2$.

(3) Find $f(x)$.

266 Find the area of the region expressed by the system of inequality $\frac{x^2}{4} + y^2 \leq 1$, $\left(1 + \frac{\sqrt{3}}{2}\right)x + y \leq 1$.

267 Let K be the curved surface obtained by rotating the parabola $y = \frac{3}{4} - x^2$ about the y -axis. Cut K by the plane H passing through the origin and forming angle 45° for the axis. Find the volume of the solid surrounded by K and H .

Note that you are not allowed to use Double Integral for the problem.

268 Find the constant numbers u, v, s, t ($s < t$) such that $\int_{-1}^1 f(x) dx = uf(s) + vf(t)$ holds for any polynomials $f(x) = ax^3 + bx^2 + cx + d$ with the degree of 3 or less than.

269 For the curve $C : y = \frac{1}{1+x^2}$, Let $A(\alpha, f(\alpha))$, $B(-\frac{1}{\alpha}, f(-\frac{1}{\alpha}))$ ($\alpha > 0$). Find the minimum area bounded by the line segments OA , OB and C , where O is the origin.

Note that you are not allowed to use the integral formula of $\frac{1}{1+x^2}$ for the problem.

270 Let $f(x)$ be the continuous function at $0 \leq x \leq 1$ such that $\int_0^1 x^k f(x) dx = 0$ for integers $k = 0, 1, \dots, n-1$ ($n \geq 1$).

(1) For all real numbers t , find the minimum value of $g(t) = \int_0^1 |x - t|^n dx$.

(2) Show the following equation for all real numbers t .

$$\int_0^1 (x - t)^n f(x) dx = \int_0^1 x^n f(x) dx$$

(3) Let M be the maximum value of the function $|f(x)|$ for $0 \leq x \leq 1$.

Show that $\left| \int_0^1 x^n f(x) dx \right| \leq \frac{M}{2^n(n+1)}$

271 For a positive constant number a , the function $f(x)$ satisfies the following equation.
 $f(x) = xe^{-\frac{x}{a}} + \frac{1}{a+1} \int_0^a f(t) dt$

(1) Express the value of $\int_0^a f(t) dt$ in terms of a .

(2) Find the maximum and minimum value of $f(x)$ for $-1 \leq x \leq 1$.

272 Find $\lim_{n \rightarrow \infty} \int_0^\pi e^x |\sin nx| dx$.

273 Find the area bounded by $0 \leq y \leq \frac{x|\sin x|}{1+|\cos x|}$, $0 \leq x \leq n\pi$ ($n = 1, 2, \dots$).

- 274** For real constant numbers a, b, c, d , consider the function $f(x) = ax^3 + bx^2 + cx + d$ such that $f(-1) = 0$, $f(1) = 0$, $f(x) \geq 1 - |x|$ for $|x| \leq 1$.

Find $f(x)$ for which $\int_{-1}^1 \{f'(x) - x\}^2 dx$ is minimized.

- 275** The line $y = mx$ has three intersection points with the curve $y = |x(x - 1)|$. Find the value of m such that the areas of two parts bounded by the line and the curve are equal.

- 276** Prove that $\int_{\frac{1}{n}}^{\pi - \frac{1}{n}} \frac{1}{\sqrt[n]{\sin x}} dx \leq \frac{3}{2}\pi$ ($n = 1, 2, 3, \dots$).

- 277** Find the function such that $f(x) = x^2 - x \left(2 \int_0^1 |f(t)| dt \right)^{\frac{1}{2}}$.

- 278** Find the value of t such that $\frac{\int_0^{\frac{\pi}{2}} (\sin x + t \cos x) dx}{\sqrt{\int_0^{\frac{\pi}{2}} (\sin x + t \cos x)^2 dx}}$ is maximized.

- 279** If the function $f(x) = 3x^2 + 2ax + b$ satisfies $\int_{-1}^1 |f(x)| dx < 2$, then show that $f(x) = 0$ has distinct real roots.

- 280** Let p, q ($p < q$) be the x coordinates of curves $x^2 + y^2 = 1$, $y \geq 0$ and $y = \frac{1}{4x}$.

(1) Find α, β such that $\cos \frac{\alpha}{2} = p$, $\cos \frac{\beta}{2} = q$ ($0 < \alpha < \pi$, $0 < \beta < \pi$).

(2) Find the area bounded by the curves.

- 281** Find the maximum and minimum values of $\int_0^\pi (a \sin x + b \cos x)^3 dx$ for $|a| \leq 1$, $|b| \leq 1$.

- 282** $g(x)$ is a differentiable function for $0 \leq x \leq \pi$ and $g'(x)$ is a continuous function for $0 \leq x \leq \pi$. Let $f(x) = g(x) \sin x$. Find $g(x)$ such that $\int_0^\pi \{f(x)\}^2 dx = \int_0^\pi \{f'(x)\}^2 dx$.

- 283** $f(x)$ is a continuous function with the periodicity of 2π and c is a positive constant number. Find $f(x)$ and c such that $\int_0^{2\pi} f(t - x) \sin t dt = cf(x)$ with $f(0) = 1$ for all real numbers x .

- 284** $f(x)$ is a continuous function which takes positive values for $x \geq 0$. Find $f(x)$ such that $\int_0^x f(t) dt = x\sqrt{f(x)}$ with $f(1) = \frac{1}{2}$.

- 285** Find the minimum value of $\int_0^1 \frac{2}{|2t-x|+3} dt$ for $0 \leq x \leq 4$.

- 286** Evaluate $\int_{-2008}^{2008} \frac{f'(x) + f'(-x)}{2008^x + 1} dx$

- 287** For constant numbers a, b , let $f(x) = e^{-x}(ax + b)$. Suppose that the curve $y = f(x)$ passes through two points $(t, 1)$, $(t + 1, 1)$. Find the area of the part bounded by the curve $y = f(x)$ and the line $y = 1$.

288 Evaluate $\int_0^1 (1 + 2008x^{2008})e^{x^{2008}} dx$.

289 Let p be a positive constant number, find $\lim_{a \rightarrow \infty} \int_0^p |x \sin(ax^2)| dx$.

- 290** Let the curve $C : y = |x^2 + 2x - 3|$ and the line l passing through the point $(-3, 0)$ with a slope m in $x - y$ plane. Suppose that C intersects to l at distinct two points other than the point $(-3, 0)$, find the value of m for which the area of the figure bounded by C and l is minimized.

- 291** Consider the parabola $C : y = x^2$ in the $x - y$ plane. Let l_1 be the tangent line of C with slope $\sqrt{3}$ and l_2 be the tangent line of C with negative slope which makes angle 45° with l_1 . Find the area bounded by C , l_1 and l_2 .

- 292** Let x, y and t be real numbers such that

$$\begin{cases} t^3 - (2y - 1)t^2 - t + 2y - 1 = 0 \\ xt - 2t + 2y = 0 \end{cases}$$

In $x - y$ plane find the area of the figure bounded by the part of curves above with $x \geq 0$ and the line $y = 2$.

- 293** Consider the parabolas $C_a : y = \frac{1}{4}x^2 - ax + a^2 + a - 2$ and $C : y = -\frac{1}{4}x^2 + 2$ for real number a in the $x - y$ plane.

(1) Find the equation of the locus of the vertex for C_a .

(2) For $a = 3$, find the slope of the two common tangent lines of C and C_a , then the intersection points of the lines.

(3) Suppose that C_a intersects with C at two distinct points. Find the maximum area of the figure bounded by C and C_a .

294 Evaluate $\int_{\ln 2}^{\ln 3} \frac{\ln(e^x + 1)}{e^{2x}} dx$.

- 295** Let $f_n(x) = x^n(1 - x)^n$, $I_n = \int_0^1 f_n(x) dx$ ($n = 1, 2, \dots$).

(1) Find a polynomial $g(x)$ such that $f'_{n+1}(x) = (n + 1)f_n(x)g(x)$.

(2) Find constant numbers A_n, B_n such that $f''_{n+2}(x) = A_n f_{n+1}(x) + B_n f_n(x)$.

(3) Find $A_n I_{n+1} + B_n I_n$.

(4) Let $J_n = (2n + 1)! I_n$. Express J_{n+1} in terms of J_n .

(5) Find I_n .

296 Let $a_n = \int_0^{\frac{\pi}{2}} (1 - \sin t)^n \sin 2t \, dt$.

(1) Find $\sum_{n=1}^{\infty} a_n$.

(2) Find $\sum_{n=1}^{\infty} \frac{a_n}{n}$.

(3) Find $\sum_{n=1}^{\infty} (n+1)(a_n - a_{n+1})$.

297 Evaluate $\int_0^{\frac{\pi}{6}} \left(\sin \frac{3}{2}x\right) \left(\cos \frac{x}{2}\right) dx$.

298 Let A be the region : $\left\{ (x, y) \mid x \geq 0, y \geq 0, \left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{1-a}\right)^{\frac{1}{2}} \leq 1 \right\}$ for $0 < a < 1$.

(1) Find the area of A .

(2) Find the maximum volume of the solid obtained by revolving A around the x axis.

299 Let $I_n(x) = \int_1^x (\ln t)^n dt$ ($x > 0$) for $n = 1, 2, 3, \dots$.

(1) Prove by mathematical induction that $I_n(x)$ is expressed by $I_n(x) = x f_n(\ln x) + C_n$ ($n \geq 1$) in terms of some polynomial $f_n(y)$ with degree n and some constant number C_n .

(2) Express the constant term of $f_n(y)$ in terms of n .

300 In Euclidean space, take the point $N(0, 0, 1)$ on the sphere S with radius 1 centered in the origin. For moving points P, Q on S such that $NP = NQ$ and $\angle PNQ = \theta$ ($0 < \theta < \frac{\pi}{2}$), consider the solid figure T in which the line segment PQ can be passed.

(1) Show that z coordinates of P, Q are equal.

(2) When P is on the plane $z = h$, express the length of PQ in terms of θ and h .

(3) Draw the outline of the cross section by cutting T by the plane $z = h$, then express the area in terms of θ and h .

(4) Pay attention to the range for which h can be valued, express the volume V of T in terms

of θ , then find the maximum V when let θ vary.

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- 301** For the positive constant number a , let D be the part surrounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the line $x + y = a$.

(1) Draw the outline of D and find the area.

(2) Find the volume of the solid by rotating D about the line $x + y = a$ as the rotation axis.

- 302** Evaluate $\int_0^\infty \frac{|x-1|}{(x+1)(x^2+1)} dx$.
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- 303** In the $x - y$ plane, find the area of the region bounded by the parameterized curve as follows.
- $$\begin{cases} x = \cos 2t \\ y = t \sin t \end{cases} \quad (0 \leq t \leq 2\pi)$$
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- 304** Let α, β be real numbers with $0 \leq \alpha \leq \beta$ and $f(x) = x^2 - (\alpha + \beta)x + \alpha\beta$ such that $\int_{-1}^1 f(x) dx = 1$. Find the maximum value of $\int_0^\alpha f(x) dx$.
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- 305** Find $\int_0^\infty (1 - \sqrt{1 - e^{-2x}}) dx$.
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- 306** For positive real numbers a, b , two graphs of the function : x^a and $\ln bx$ have a tangency of point.

(1) Let (s, t) be the tangency of point, express this in terms of a , then express b as the function of a .

(2) For h with $0 < h < s$, denote the area as $A(h)$ of the domain bounded by the line $x = h$ and two curves $y = x^a, y = \ln bx$.

Express $\lim_{h \rightarrow 0} A(h)$ in terms of a .

- 307** For real numbers a, b, c , let $f(x) = ax^2 + bx + c$.
Prove that : $\int_{-1}^1 (1 - x^2) \{f'(x)\}^2 dx \leq 6 \int_{-1}^1 \{f(x)\}^2 dx$.
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- 308** Let a be a positive constant number. For a positive integer n , define a function $I_n(t)$ by $I_n(t) = \int_0^t x^n e^{-ax} dx$. Answer the following questions.

Note that you may use $\lim_{t \rightarrow \infty} t^n e^{-at} = 0$ without proof.

(1) Evaluate $I_1(t)$.

(2) Find the relation of $I_{n+1}(t)$, $I_n(t)$.

(3) Prove that there exists $\lim_{t \rightarrow \infty} I_n(t)$ for all natural number n by using mathematical induction.

(4) Find $\lim_{t \rightarrow \infty} I_n(t)$.

309 (1) Calculate the indefinite integral :

$$\int e^{-x} \sin 2x \, dx.$$

(2) Evaluate the definite integral :

$$\int_0^\pi e^{-x} |\sin 2x| \, dx.$$

310 Define the function $f(x)$ as : $f(x) = \begin{cases} \frac{\ln(1+x)}{1+x} & (x \geq 0) \\ \frac{\ln(1-x)}{1-x} & (-1 \leq x < 0) \end{cases}$

1. Examine the variation of $f(x)$ and find the maximum and minimum value of $f(x)$.

2. Find the value of a for which $\int_{-1}^{e-1} |f(x) - a| \, dx$ is minimized for $0 \leq a \leq \frac{1}{e}$.

311 Prove the following inequality.

$$\int_0^1 2008^{x^{2008}} \, dx \geq \left(1 - \frac{1}{2008^{2008}}\right) \frac{1}{\ln 2008}.$$

312 Let a, b be positive real numbers. For a real number t , denote by $d(t)$ the distance between the origin and the line $(ae^t)x + (be^{-t})y = 1$.

Let a, b vary with $ab = 1$, find the minimum value of $\int_0^1 \frac{1}{d(t)^2} \, dt$.

313 Find the area bounded by the graph of $y = \sqrt[3]{x + \sqrt{x^2 + 1}} + \sqrt[3]{x - \sqrt{x^2 + 1}}$, the line $x - y - 1 = 0$ and the x axis.

314 Evaluate $\int_{\frac{\pi}{2}}^\pi x^{\sin x} (1 + x \cos x \cdot \ln x + \sin x) \, dx$.

315 Evaluate $\int_0^1 \frac{(x^2 - 3x + 1)e^x - (x - 1)e^{2x} - x}{(x + e^x)^3} \, dx$.

316 Evaluate $\int_0^1 \frac{e^{2x} - e^{-2x} + 4x}{(e^x + e^{-x})^2} \, dx$.

317 Evaluate $\int_0^1 \frac{x + \sqrt{x} - 1}{(\sqrt{x} + 1)^2} \, dx$.

318 Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1 + \cot x}{e^x \sin x} \, dx$.

319 Evaluate $\int_{-1}^1 \frac{xe^x + e^x + 1}{x^2(e^x + 1)^2} dx$.

320 Evaluate $\int_0^\pi e^{x \sin x} (x^2 \cos x + x \sin x + 1) dx$.

321 Evaluate $\int_{-1}^1 \frac{x^2 e^x - x^2 e^{2x} + 2e^x + 2}{(e^x + 1)^2} dx$.

322 Evaluate $\int_1^e \frac{(x^4 - 4x^3 + 9x^2 - 6x + 8)e^x}{x^3} dx$.

323 Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\sqrt{\frac{\sin x}{x}} + \sqrt{\frac{x}{\sin x}} \cos x \right) dx$.

324 Evaluate $\int_1^e \left(\frac{1}{\sqrt{x \ln x}} + \sqrt{\frac{\ln x}{x}} \right) dx$.

325 Prove that $\frac{1}{3} < \int_0^1 x^{(\sin x + \cos x)^2} dx < \frac{1}{2}$.

326 Evaluate $\int_1^e \frac{(x+1)(x+2-\ln x)}{(x+\ln x)^2} dx$.

327 Let a, b be real numbers and C be the graph of the function $y = e^{a+bx^2}$.

(1) Find the values of a, b such that C passes through the point $P(1, 1)$ and the slope of the tangent line of C at P is -2 .

(2) For the values of a, b found in (1), find the volume of the solid generated by the revolution of the part which lies in the right side for the y axis in the figure bounded by the parabola $y = x^2$ and the curve C about the y axis.

328 Let t be a negative real number and D be the part bounded by the curve $y = 2^{2x+2t}$, the curve $y = 2^{x+3t}$ and the y axis.

(1) Find the volume $V(t)$ generated by rotation of D about the x axis.

(2) When t moves in the range of negative real number, find the maximum value of $V(t)$.

329 Let $f(x), g(x)$ be continuous functions defined in the range of $0 \leq x \leq 1$.

(1) Find $f(x)$ such that $f(x) = \int_0^1 e^{x+t} f(t) dt$.

(2) Find $g(x)$ such that $g(x) = \int_0^1 e^{x+t} g(t) dt + x$.

330 Find all real numbers x such that $\int_0^x t^2 \sin(x-t) dt = x^2$.

331 For $a \in \mathbb{R}$, find the minimum value of $\int_0^{\frac{\pi}{2}} \left| \frac{\sin 2x}{1+\sin^2 x} - a \cos x \right| dx$.

332 Let $f(x)$ be a function such that $1 - f(x) = f(1-x)$ for $0 \leq x \leq 1$.

Evaluate $\int_0^1 f(x) dx$.

333 Find the functions $f(x)$, $g(x)$ such that $f(x) + \int_0^x g(t) dt = \sin x(\cos x - \sin x)$, $\{f'(x)\}^2 + \{g(x)\}^2 = 1$.

334 Evaluate $\int_0^1 \frac{7x^3+23x^2+21x+15}{(x^2+1)(x+1)^2} dx$.

335 For $t > 0$, prove that $\frac{1}{2t} \int_t^{3t} x \ln x dx \geq 2t \ln 2t$.

336 Let $P_n = \sqrt[n]{\frac{(3n)!}{(2n)!}}$ ($n = 1, 2, \dots$).
Find the following limits.

(1) $\lim_{n \rightarrow \infty} \frac{P_n}{n}$.

(2) $\lim_{n \rightarrow \infty} \left(\frac{n+2}{n}\right)^{P_n}$.

337 Find $\lim_{n \rightarrow \infty} \int_0^{\frac{\pi}{2}} \sum_{k=1}^n \sin kx \sin x dx$.

338 Given a parameterized curve $C : x = e^t - e^{-t}$, $y = e^{3t} + e^{-3t}$.

Find the area bounded by the curve C , the x axis and two lines $x = \pm 1$.

339 Find the minimum area of the part bounded by the parabola $y = a^3x^2 - a^4x$ ($a > 0$) and the line $y = x$.

340 Find the continuous function $f(x)$ such that $xf(x) - \int_0^x f(t) dt = x + \ln(\sqrt{x^2+1} - x)$ with $f(0) = \ln 2$.

341 Let c be a constant number. Find the real value of x such that $\int_0^c |t-x| dt = \int_0^x |t-c| dt$.

342 Prove the following inequality.

$$\left(\int_0^1 (x^2 + px + q) dx\right)^2 \leq \int_0^1 (x^2 + px + q)^2 dx - \frac{1}{180}$$

- 343** Find all continuous positive functions $f(x)$, for $0 \leq x \leq 1$, such that

$$\int_0^1 f(x) dx = 1$$

$$\int_0^1 x f(x) dx = \alpha$$

$$\int_0^1 x^2 f(x) dx = \alpha^2$$

where α is given real number.

- 344** Find the value of k ($0 < k < 5$) such that $\int_0^\infty \frac{x^k}{2+4x+3x^2+5x^3+3x^4+4x^5+2x^6} dx$ is minimal.

- 345** Given a continuous function $f(x)$ such that $\int_0^{2\pi} f(x) dx = 0$.

Let $S(x) = A_0 + A_1 \cos x + B_1 \sin x$, find constant numbers A_0 , A_1 and B_1 for which $\int_0^{2\pi} \{f(x) - S(x)\}^2 dx$ is minimized.

- 346** Suppose that two curves $C_1 : y = x^3 - x$, $C_2 : y = (x-a)^3 - (x-a)$ have two intersection points.

Find the maximum area of the figure bounded by C_1 , C_2 .

- 347** For real positive numbers a , find the minimum value of $\int_0^\pi \left(\frac{x}{a} + a \sin x\right)^2 dx$.

- 348** Find $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^{\frac{1}{n^2}} \left(\frac{n+2}{n}\right)^{\frac{2}{n^2}} \left(\frac{n+3}{n}\right)^{\frac{3}{n^2}} \dots \left(\frac{n+n}{n}\right)^{\frac{n}{n^2}}$ ($n = 1, 2, 3, \dots$).

- 349** For real numbers a , b ($ab \neq 0$), evaluate $\int_0^\pi (a \sin x + b \cos x)^6 dx$.

- 350** Evaluate $\int_0^1 (1-x^2)^{\frac{5}{2}} dx$.

- 351** Find the positive value of k for which $\int_0^{\frac{\pi}{2}} |\cos x - kx| dx$ is minimized.

- 352** Prove the following inequality.

$$1 \leq \lim_{n \rightarrow \infty} \int_{\frac{1}{n}}^{\frac{\pi}{2}} \frac{\pi \sin x}{x(\pi - x)} dx < 3$$

- 353** Consider a parabola $C : y = \frac{1}{4}x^2$ and the point $F(0, 1)$. For the origin O , take n points on the parabola C , $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, $\dots$, $A_n(x_n, y_n)$ such that $x_k > 0$ and $\angle OFA_k = \frac{k\pi}{2n}$ ($k = 1, 2, 3, \dots, n$). Find $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n FA_k$.

354 Evaluate $\int_{-2}^2 x^4 \sqrt{4-x^2} dx$.

355 For a positive number n , find $\lim_{n \rightarrow \infty} n^2 \int_{e^{-\frac{1}{n}}}^1 x^n \ln x dx$.

356 A continuous function $f(x)$ satisfies that $\frac{d}{dx} \left(\int_0^x f(x+t) dt \right) = 0$. Find the necessary and sufficient condition such that $\sum_{n=1}^{\infty} f(2^n) = 1$.

357 For $0 < a < 1$, let $S(a)$ is the area of the figure bounded by three curves $y = e^x$, $y = e^{\frac{1+a}{1-a}x}$ and $y = e^{2-x}$.
Find $\lim_{a \rightarrow 0} \frac{S(a)}{a}$.

358 Let a be a positive constant number. Given a positive integer n , take an integer m such that $m \leq \frac{na}{\pi} < m+1$.
Find $\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^{m\pi} \left(a - \frac{t}{n}\right) |\sin t| dt$.

359 Evaluate $\int_{-1}^1 \frac{2x^4 - x^3 - 2x^2 + 1}{x^3 - x + \sqrt{1-x^2}} dx$.

360 Evaluate $\int_{-1}^1 (1+x)^{\frac{1}{2}} (1-x)^{\frac{3}{2}} dx$.

361 Find the following indefinite integrals.

(1) $\int \frac{\sin x}{3 + \sin^2 x} dx$.

(2) $\int e^{2x+e^x} dx$.

362 Evaluate the following definite integrals.

(1) $\int_0^{\frac{\pi}{2}} x \cos x dx$.

(2) $\int_1^2 \frac{1}{x(x+1)} dx$.

(3) $\int_0^1 \frac{1}{1+e^x} dx$.

(4) $\int_0^{\frac{\pi}{4}} \cos 2x \cos 3x dx$.

363 For $t \geq 0$, let $a_n = \int_0^t e^{nx} dx$ ($n = 0, 1, 2, 3$).

(1) Show that $a_3 - 3a_2 + 3a_1 - a_0 \geq 0$.

(2) Show that $e^t a_0 + (e^t - 1)a_1 - a_2 \geq 0$.

364 Evaluate $\int_0^1 x^2(x-1)^2 e^{2x} dx$.

365 Let a, b be positive constant numbers.

Evaluate $\int_0^{\frac{\pi}{2}} \frac{a \cos x - b \sin x}{b \sin x + a \cos x} dx$.

366 (1) Determine the constant numbers a, b, c, p, q, r, s such that the following equation is equality.

$$4x = \{a(x+1)^2 + b(x+1) + c\}(x^2+1)^2 + \{(px+q)(x^2+1) + (rx+s)\}(x+1)^3.$$

(2) Evaluate the following definite integrals.

$$\begin{aligned} (a) \int_0^1 \frac{dx}{x+1} \quad (b) \int_0^1 \frac{dx}{(x+1)^2} \quad (c) \int_0^1 \frac{dx}{(x+1)^3} \\ (d) \int_0^1 \frac{x}{x^2+1} dx \quad (e) \int_0^1 \frac{dx}{x^2+1} \quad (f) \int_0^1 \frac{x}{(x^2+1)^2} dx \\ (g) \int_0^1 \frac{dx}{(x^2+1)^2} \quad (h) \int_0^1 \frac{4x}{(x+1)^3(x^2+1)^2} dx \end{aligned}$$

367 Let $f(x) = x^2 + \int_0^1 x f(t) dt + \int_{-1}^2 f(t) dt$.

Evaluate $\int_0^{\frac{\pi}{2}} f(x + \sin x) dx$.

368 For a real number a , evaluate $\int_0^a |x(x-a^2)| dx$.

369 Calculate

$$\int_{e^{e^e}}^{e^{e^{e^e}}} \frac{1}{x \ln x \ln(\ln x) \ln(\ln(\ln x))} dx.$$

370 Calculate

$$\int_0^{\pi/3} \frac{\cos^2 x + 1}{\cos x \sqrt{\cos x}} dx.$$

371 Calculate

$$\int_{-1}^1 \frac{1}{(1+e^x)(1+x^2)} dx.$$

372 Evaluate

$$\int_0^1 e^{x+e^{x+e^{x+e^x}}} dx.$$

373 Evaluate

$$\int_0^{\frac{\pi}{2}} e^x \left\{ \cos(\sin x) \cos^2 \frac{x}{2} + \sin(\sin x) \sin^2 \frac{x}{2} \right\} dx.$$

374 Let $n \geq 2$ be positive integers.

(1) Prove that $n \ln n - n + 1 < \sum_{k=1}^n \ln k < (n+1) \ln n - n + 1$.

(2) Find $\lim_{n \rightarrow \infty} (n!)^{\frac{1}{n \ln n}}$.

375 Prove the following inequality.

$$\frac{1}{n} < \int_0^{\frac{\pi}{2}} \frac{1}{(1+\cos x)^n} dx < \frac{n+5}{n(n+1)} \quad (n = 2, 3, \dots).$$

376 Evaluate $\int_0^{\frac{\pi}{4}} \frac{1+2\sin x - \sin^2 x}{(1+x)\cos^4 x} dx$.

377 The line $y = mx$ has 3 intersection points with the curve $y = |x(x-1)|$. Find the value of m such that the area of the 2 regions bounded by the line and the curve are equal.

378 Evaluate $\int_0^{\pi} x |\sin nx| dx$ ($n = 1, 2, \dots$).

379 Let α, r be real numbers such that $r > 1$, $r \neq 3$, $r \neq 4$.

Find the values of α, r such that $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n^\alpha}{(n+k)^r} = \frac{r-3}{(r-1)(r-4)}$.

380 Find $\int_{-1}^1 \binom{n}{k} (1+x)^{n-k} (1-x)^k dx$ ($k = 0, 1, 2, \dots, n$).

381 A function $f(x)$ is defined as follows for $x \geq 0$.

$$f(x) = \sin\left(\frac{n\pi}{4}\right) \sin x \quad (n\pi \leq x < (n+1)\pi) \quad (n = 0, 1, 2, \dots).$$

Evaluate $\int_0^{100\pi} f(x) dx$.

382 (1) For $a > 0$, $b \geq 0$, Compare

$$\int_b^{b+1} \frac{dx}{\sqrt{x+a}}, \quad \frac{1}{\sqrt{a+b}}, \quad \frac{1}{\sqrt{a+b+1}}.$$

(2) Find $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{n^2+k}}$.

383 For a positive integer m , evaluate $\int_0^{\frac{\pi}{2}} \cos^m x \sin 2mx dx$.

384 Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sin x + \cos x) \sqrt{\frac{e^x}{\sin x}} dx$.

385 Evaluate $\int_0^{\frac{\pi^2}{4}} (2 \sin \sqrt{x} + \sqrt{x} \cos \sqrt{x}) dx$.

386 For $a = 2007 * 2009$, evaluate $\int_0^1 \frac{6x+5a\sqrt{x}}{4\sqrt{x+a\sqrt{x}}} dx$.

387 Let l_1, l_2 be the tangent and normal line respectively at the point $(p, \ln(p+1))$ on the curve $C: y = \ln(x+1)$. Denote by T_i ($i = 1, 2$) the areas bounded by l_i ($i = 1, 2$), C and the y axis respectively. Find the limit $\lim_{p \rightarrow 0} \frac{T_2}{T_1}$.

388 For $f(x) = \sin^3 x$, let $p(x)$ be quadratic polynomial. Evaluate $\int_0^{2\pi} p(x) f''(x) dx$.

389 Find the values of $t \in [0, 2]$ for which $\int_{t-3}^{2t} 2x^2 dx$ is maximal and minimal.

390 Find the polynomials $f(x), g(x)$ such that:

$$\frac{1}{\pi} \int_0^{\frac{1}{2}} \frac{tf'(t)-6g(t)}{\sqrt{1-t^2}} dt = f(x) - g(x) + x$$

$$\frac{6}{\pi} \int_0^{\frac{1}{2}} \frac{8f(t)-5g'(t)}{\sqrt{1-t^2}} dt = 2f(x) - 3g(x) - x^2 + 2x.$$

391 Evaluate $\int_0^{\frac{\pi}{2}} \sin |2x - a| dx$ for a real number $a \in [0, \pi]$.

392 Evaluate $\int_{-1}^1 \frac{|x|}{1+e^x} dx$.

393 Let $V(a)$ be the volume of the solid obtained by revolving the region bounded by the curve $y = x\sqrt{x \sin ax}$ ($0 \leq x \leq \frac{\pi}{a}$) and x axis around the x axis.

For $a > 0$, find the minimum value of $V(a) + V\left(\frac{1}{a}\right)$.

394 Find $\lim_{x \rightarrow 0} \frac{1}{x} \int_{-x}^x (t \sin 2006t + 2007t + 1004) dt$.

395 3 points $O(0, 0), P(a, a^2), Q(-b, b^2)$ ($a > 0, b > 0$) are on the parabola $y = x^2$. Let S_1 be the area bounded by the line PQ and the parabola and let S_2 be the area of the triangle OPQ .

Find the minimum value of $\frac{S_1}{S_2}$.