NMO 2012

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Day 1

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1 A five-digit positive integer $abcde_{10}$ ($a \neq 0$) is said to be a *range* if its digits satisfy the inequalities $a < b > c < d > e$. For example, 37452 is a range. How many ranges are there?

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2 In triangle $[ABC]$, the bissector of the angle $\angle BAC$ intersects the side $[BC]$ at D. Suppose that $\overline{AD} = \overline{CD}$. Find the lengths $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ that minimize the perimeter of $[ABC]$, given that all the sides of the triangles $[ABC]$ and $[ADC]$ have integer lengths.

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3 Helena and Luis are going to play a game with two bags with marbles. They play alternately and on each turn they can do one and only one of the following moves:

Take out a marble from one bag.
Take out a marble from each bag.
Take out a marble from one bag and then put it into the other bag.

The player who leaves both bags empty wins the game.

Before starting the game, Helena counted out the marbles of each bag and said to Luis: ”You may start!”, while she thought ”I will certainly win...”. What are the possible distributions of the marbles in the bags?

Day 2

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1 Find the number of positive integers n such that $1 \leq n \leq 1000$ and n is divisible by $\lfloor \sqrt[3]{n} \rfloor$.

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2 Let $[ABC]$ be a triangle. Points D, E, F and G are such E and F are on the lines AC and BC, respectively, and $[ACFG]$ and $[BCED]$ are rhombus. Lines AC and BG meet at H; lines BC and AD meet at I; lines AI and BH meet at J. Prove that $[JICH]$ and $[ABJ]$ have equal area.

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3 Isabel wants to partition the set $\mathbb{N}$ of the positive integers into n disjoint sets $A_1, A_2, \ldots, A_n$. Suppose that for each i with $1 \leq i \leq n$, given any positive integers $r, s \in A_i$ with $r \neq s$, we have $r + s \in A_i$. If $|A_j| = 1$ for some j, find the greatest positive integer that may belong to A_j.
