Kazakhstan National Olympiad 2005

www.artofproblemsolving.com/community/c4039

by rightways, qwert159, birzhan

Day 1

---

1 Does there exist a solution in real numbers of the system of equations

$$\begin{cases} (x-y)(z-t)(z-x)(z-t)^2 = A, \\ (y-z)(t-x)(t-y)(x-z)^2 = B, \\ (x-z)(y-t)(z-t)(y-z)^2 = C, \end{cases}$$

when
a) $A = 2, B = 8, C = 6;$

b) $A = 2, B = 6, C = 8.$?

---

2 Prove that

$$ab + bc + ca \geq 2(a+b+c)$$

where a, b, c are positive reals such that $a + b + c + 2 = abc$.

---

3 Call a set of points in the plane $good$ if any three points of the set are the vertices of an isosceles triangle or if they are collinear. Determine all $a)$ 6-element $good$ sets $b)$ 7-element $good$ sets.

---

4 Find all functions $f : \mathbb{R} \to \mathbb{R}$, satisfying the condition

$f(f(x) + x + y) = 2x + f(y)$

for any real x and y.

Day 2

---

1 Solve equation

$$2^{\frac{1}{2} - 2|x|} = \left|\tan x + \frac{1}{2}\right| + \left|\tan x - \frac{1}{2}\right|$$

---

2 The line parallel to side AC of a right triangle ABC $(\angle C = 90^\circ)$ intersects sides AB and BC at M and N, respectively, so that the $CN/BN = AC/BC = 2$. Let O be the intersection point of the segments AN and CM and K be a point on the segment ON such that $MO + OK = KN$. The perpendicular line to AN at point K and the bisector of triangle ABC of $\angle B$ meet at point T. Find the angle $\angle MTB$.

---

3 Exactly one number from the set $\{-1, 0, 1\}$ is written in each unit cell of a 2005×2005 table, so that the sum of all the entries is 0. Prove that there exist two rows and two columns of the table, such that the sum of the four numbers written at the intersections of these rows and columns is equal to 0.

---

4 Find all polynomials $P(x)$ with real coefficients such that for every positive integer n there exists a rational r with $P(r) = n$.
