Kazakhstan National Olympiad 2007

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by rightways

Day 1

1 Zeros of a fourth-degree polynomial $f(x)$ form an arithmetic progression. Prove that the zeros of $f'(x)$ also form an arithmetic progression.

2 Let ABC be an isosceles triangle with $AC = BC$ and I is the center of the inscribed circle. The point P lies on the circle circumscribed about the triangle AIB and lies inside the triangle ABC. Straight lines passing through point P parallel to CA and CB intersect AB at points D and E, respectively. The line through P which is parallel to AB intersects CA and CB at points F and G, respectively. Prove that the lines DF and EG meet at the circumcircle of triangle ABC.

3 Solve in prime numbers the equation $p(p+1)+q(q+1) = r(r+1)$.

4 Several identical square sheets of paper are laid out on a rectangular table so that their sides are parallel to the edges of the table (sheets may overlap). Prove that you can stick a few pins in such a way that each sheet will be attached to the table exactly by one pin.

Day 2

1 Convex quadrilateral $ABCD$ with AB not equal to DC is inscribed in a circle. Let $AKDL$ and $CMBN$ be rhombs with same side of a. Prove that the points K, L, M, N lie on a circle.

2 Each cell of a 100 x 100 board is painted in one of 100 different colors so that there are exactly 100 cells of each color. Prove that there is a row or column in which there are at least 10 cells of different colors.

3 Let p be a prime such that $2^{p-1} \equiv 1 \pmod{p^2}$. Show that $(p-1)(p!+2^n)$ has at least three distinct prime divisors for each $n \in \mathbb{N}$.

4 Find all functions $f:\mathbb{R}\rightarrow\mathbb{R}$, satisfying the condition

$$f(xf(y)+f(x)) = 2f(x)+xy$$

for any real x and y.