Kazakhstan National Olympiad 2008

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Day 1

1 Let F_n be a set of all possible connected figures, that consist of n unit cells. For each element f_n of this set, let $S(f_n)$ be the area of that minimal rectangle that covers f_n and each side of the rectangle is parallel to the corresponding side of the cell. Find $max(S(f_n))$,where $f_n \in F_n$?

Remark: Two cells are called connected if they have a common edge.

2 Suppose that B_1 is the midpoint of the arc AC, containing B, in the circumcircle of $\triangle ABC$, and let I_b be the B-excircle's center. Assume that the external angle bisector of $\angle ABC$ intersects AC at B_2. Prove that B_2I is perpendicular to B_1I_B, where I is the incenter of $\triangle ABC$.

3 Let $f(x,y,z)$ be the polynomial with integer coefficients. Suppose that for all reals x,y,z the following equation holds:

$$f(x,y,z) = -f(x,z,y) = -f(y,x,z) = -f(z,y,x)$$

Prove that if $a,b,c \in \mathbb{Z}$ then $f(a,b,c)$ takes an even value

Day 2

1 Find all integer solutions $(a_1,a_2,\dots,a_{2008})$ of the following equation:
$(2008-a_1)^2+(a_1-a_2)^2+\cdots+(a_{2007}-a_{2008})^2+a_{2008}^2=2008$

2 Let $\triangle ABC$ be a triangle and let K be some point on the side AB, so that the tangent line from K to the incircle of $\triangle ABC$ intersects the ray AC at L. Assume that ω is tangent to sides AB and AC, and to the circumcircle of $\triangle AKL$. Prove that ω is tangent to the circumcircle of $\triangle ABC$ as well.

3 Find the maximum number of planes in the space, such there are 6 points, that satisfy to the following conditions:
1.Each plane contains at least 4 of them
2.No four points are collinear.