Kazakhstan National Olympiad 2010

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Day 9

1 Triangle ABC is given. Circle ω passes through B, touch AC in D and intersect sides AB and BC at P and Q respectively. Line PQ intersect BD and AC at M and N respectively. Prove that ω, circumcircle of DMN and circle, touching PQ in M and passes through B, intersects in one point.

2 Exactly $4n$ numbers in set $A = \{1, 2, 3, ..., 6n\}$ of natural numbers painted in red, all other in blue.
Proved that exist $3n$ consecutive natural numbers from A, exactly $2n$ of which numbers is red.

3 Positive real A is given. Find maximum value of M for which inequality

$\frac{1}{x} + \frac{1}{y} + \frac{A}{x+y} \geq \frac{M}{\sqrt{xy}}$

holds for all $x, y > 0$

4 Let x- minimal root of equation $x^2 - 4x + 2 = 0$.
Find two first digits of number $\{x + x^2 + ... + x^{20}\}$ after 0, where $\{a\}$- fractional part of a.

5 Arbitrary triangle ABC is given (with $AB < BC$). Let M - midpoint of AC, N- midpoint of arc AC of circumcircle ABC, which is contains point B. Let I - in-center of ABC. Proved, that $\angle IMA = \angle INB$

6 Let numbers $1, 2, 3, ..., 2010$ stand in a row at random. Consider row, obtain by next rule:
For any number we sum it and it's number in a row (For example for row $(2, 7, 4)$ we consider a row $(2 + 1; 7 + 2; 4 + 3) = (3; 9; 7)$);
Proved, that in resulting row we can found two equals numbers, or two numbers, which is differ by 2010

Day 10

1 Triangle ABC is given. Consider ellipse Ω_1, passes through C with focuses in A and B. Similarly define ellipses Ω_2, Ω_3 with focuses B, C and C, A respectively. Prove, that if all ellipses have common point D then A, B, C, D lies on the circle.

Ellipse with focuses X, Y, passes through Z- locus of point T, such that $XT + YT = XZ + YZ$

2 On sides of convex quadrilateral $ABCD$ on external side constructed equilateral triangles ABK, BCL, CDM, DAN. Let P, Q- midpoints of BL, AN respectively and X- circumcenter of CMD.
Prove, that PQ perpendicular to KX

3 Call $A \in \mathbb{N}^0$ be $numberofyear$ if all digits of A equals $0, 1$ or 2 (in decimal representation).
Prove that exist infinity $N \in \mathbb{N}$, such that N can't presented as $A^2 + B$ where $A \in \mathbb{N}^0; B$-$numberofyear$.

4 It is given that for some $n \in \mathbb{N}$ there exists a natural number a, such that $a^{n-1} \equiv 1 \pmod{n}$ and that for any prime divisor p of $n-1$ we have $a^{\frac{n-1}{p}} \not\equiv 1 \pmod{n}$.
Prove that n is a prime.

5 Let $n \geq 2$ be an integer. Define $x_i = 1$ or -1 for every $i = 1, 2, 3, \cdots, n$.

Call an operation *adhesion*, if it changes the string $(x_1, x_2, \cdots, x_n)$ to $(x_1x_2, x_2x_3, \cdots, x_{n-1}x_n, x_nx_1)$
.

Find all integers $n \geq 2$ such that the string $(x_1, x_2, \cdots, x_n)$ changes to $(1, 1, \cdots, 1)$ after finitely *adhesion* operations.

6 Let $ABCD$ be convex quadrilateral, such that exist M, N inside $ABCD$ for which $\angle NAD = \angle MAB; \angle NBC = \angle MBA; \angle MCB = \angle NCD; \angle NDA = \angle MDC$
Prove, that $S_{ABM} + S_{ABN} + S_{CDM} + S_{CDN} = S_{BCM} + S_{BCN} + S_{ADM} + S_{ADN}$, where S_{XYZ}-area of triangle XYZ

Day 11

1 It is given that for some $n \in \mathbb{N}$ there exists a natural number a, such that $a^{n-1} \equiv 1 \pmod{n}$ and that for any prime divisor p of $n-1$ we have $a^{\frac{n-1}{p}} \not\equiv 1 \pmod{n}$.
Prove that n is a prime.

2 Let $n \geq 2$ be an integer. Define $x_i = 1$ or -1 for every $i = 1, 2, 3, \cdots, n$.

Call an operation *adhesion*, if it changes the string $(x_1, x_2, \cdots, x_n)$ to $(x_1x_2, x_2x_3, \cdots, x_{n-1}x_n, x_nx_1)$
.

Find all integers $n \geq 2$ such that the string $(x_1, x_2, \cdots, x_n)$ changes to $(1, 1, \cdots, 1)$ after finitely *adhesion* operations.

3 Let $ABCD$ be convex quadrilateral, such that exist M, N inside $ABCD$ for which $\angle NAD = \angle MAB; \angle NBC = \angle MBA; \angle MCB = \angle NCD; \angle NDA = \angle MDC$

Prove, that $S_{ABM}+S_{ABN}+S_{CDM}+S_{CDN}=S_{BCM}+S_{BCN}+S_{ADM}+S_{ADN}$, where S_{XYZ}-area of triangle XYZ

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4 For $x; y \geq 0$ prove the inequality:

$\sqrt{x^2-x+1}\sqrt{y^2-y+1}+\sqrt{x^2+x+1}\sqrt{y^2+y+1} \geq 2(x+y)$

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5 Let O be the circumcircle of acute triangle ABC, AD-altitude of ABC ($D \in BC$), $AD \cap CO = E$, M-midpoint of AE, F-feet of perpendicular from C to AO.
Proved that point of intersection OM and BC lies on circumcircle of triangle BOF

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Prove that exist infinity $N \in \mathbb{N}$, such that N can't presented as A^2+B where $A \in \mathbb{N}^0; B$-$numberofyear$.
