Kazakhstan National Olympiad 2011

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by ts0_9

–	Grade 9
1	The quadrilateral $ABCD$ is circumscribed about the circle, touches the sides AB, BC, CD, DA in the points K, L, M, N, respectively. Let P, Q, R, S midpoints of the sides KL, LM, MN, NK. Prove that $PR = QS$ if and only if $ABCD$ is inscribed.
2	Determine the smallest possible number $n > 1$ such that there exist positive integers $a_1, a_2, \ldots, a_n$ for which ${a_1}^2 + \cdots + {a_n}^2 \mid (a_1 + \cdots + a_n)^2 - 1$.
3	In some cells of a rectangular table $m \times n (m, n > 1)$ is one checker. $Baby$ cut along the lines of the grid this table so that it is split into two equal parts, with the number of pieces on each side were the same. $Carlson$ changed the arrangement of checkers on the board (and on each side of the cage is still worth no more than one pieces). Prove that the $Baby$ may again cut the board into two equal parts containing an equal number of pieces
4	We write in order of increasing number of 1 and all positive integers,which the sum of digits is divisible by 5. Obtain a sequence of $1, 5, 14, 19...$ Prove that the n-th term of the sequence is less than $5n$.
5	Given a non-degenerate triangle ABC, let A_1, B_1, C_1 be the point of tangency of the incircle with the sides BC, AC, AB. Let Q and L be the intersection of the segment AA_1 with the incircle and the segment B_1C_1 respectively. Let M be the midpoint of B_1C_1. Let T be the point of intersection of BC and B_1C_1. Let P be the foot of the perpendicular from the point L on the line AT. Prove that the points A_1, M, Q, P lie on a circle.
6	Given a positive integer n. One of the roots of a quadratic equation $x^2 - ax + 2n = 0$ is equal to $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$. Prove that $2\sqrt{2n} \le a \le 3\sqrt{n}$
–	Grade 10
1	Inscribed in a triangle ABC with the center of the circle I touch the sides AB and AC at points C_1 and B_1, respectively. The point M divides the segment C_1B_1 in a 3:1 ratio, measured from C_1. N - the midpoint of AC. Prove that the points I, M, B_1, N lie on a circle, if you know that $AC = 3(BC - AB)$.
2	Given a positive integer n. Prove the inequality $\sum\limits_{i=1}^{n} \frac{1}{i(i+1)(i+2)(i+3)(i+4)} < \frac{1}{96}$

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4 Prove that there are infinitely many natural numbers, the arithmetic mean and geometric mean of the divisors which are both integers.

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6 Determine all pairs of positive real numbers (a, b) for which there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying for all positive real numbers x the equation $f(f(x)) = af(x) - bx$

– Grade 11

1 Given a real number $a > 0$. How many positive real solutions of the equation is $a^x = x^a$

2 Let w-circumcircle of triangle ABC with an obtuse angle C and C'symmetric point of point C with respect to AB. M midpoint of AB. $C'M$ intersects w at N (C' between M and N). Let BC' second crossing point w in F, and AC' again crosses the w at point E. K-midpoint EF. Prove that the lines AB, CN and$KC'$are concurrent.

3 Given are the odd integers $m > 1$, k, and a prime p such that $p > mk + 1$. Prove that $p^2 \mid \binom{k}{k}^m + \binom{k+1}{k}^m + \cdots + \binom{p-1}{k}^m$.

4 Prove that there are infinitely many natural numbers, the arithmetic mean and geometric mean of the divisors which are both integers.

5 On the table lay a pencil, sharpened at one end. The student can rotate the pencil around one of its ends at 45° clockwise or counterclockwise. Can the student, after a few turns of the pencil, go back to the starting position so that the sharpened end and the not sharpened are reversed?

6 We call a square table of a binary, if at each cell is written a single number 0 or 1. The binary table is called regular if each row and each column exactly two units. Determine the number of regular size tables $n \times n$ ($n > 1$ - given a fixed positive integer). (We can assume that the rows and columns of the tables are numbered: the cases of coincidence in turn, reflect, and so considered different).