Kazakhstan National Olympiad 2012

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by ts0_9

–	Grade level 9
Day 1	
1	Solve the equation $p+\sqrt{q^2+r}=\sqrt{s^2+t}$ in prime numbers.
2	Given two circles k_1 and k_2 with centers O_1 and O_2 that intersect at the points A and B.Passes through A two lines that intersect the circle k_1 at the points $N_1$and M_1, and the circle k_2 at the points N_2 and M_2 (points A, N_1, M_1 in colinear). Denote the midpoints of the segments N_1N_2 and $M_1M_{2]}$, through N and M.Prove that: $a)$ Points M, N, A and B lie on a circle $b)$The center of the circle passing through M, N, A and B lies in the middle of the segment O_1O_2
3	Let $a, b, c, d > 0$ for which the following conditions:: $a)$ $(a-c)(b-d)=-4$ $b)$ $\frac{a+c}{2} \geq \frac{a^2+b^2+c^2+d^2}{a+b+c+d}$ Find the minimum of expression $a+c$
Day 2	
1	Do there exist a infinite sequence of positive integers (a_n) ,such that for any $n \geq 1$ the relation $a_{n+2}=\sqrt{a_{n+1}}+a_n$?
2	Given an inscribed quadrilateral $ABCD$, which marked the midpoints of the points M, N, P, Q in this order. Let diagonals AC and BD intersect at point O. Prove that the triangle OMN, ONP, OPQ, OQ have the same radius of the circles
3	The cell of a $(2m+1) \times (2n+1)$ board are painted in two colors - white and black. The unit cell of a row (column) is called *dominant* on the row (the column) if more than half of the cells that row (column) have the same color as this cell. Prove that at least $m+n-1$ cells on the board are dominant in both their row and column.
–	Grade level 10
Day 1	
1	For a positive reals $x_1, ..., x_n$ prove inequlity: $\frac{1}{x_1+1}+...+\frac{1}{x_n+1} \leq \frac{n}{1+\frac{n}{\frac{1}{x_1}+...+\frac{1}{x_n}}}$
2	Let $ABCD$ be an inscribed quadrilateral, in which $\angle BAD < 90$. On the rays AB and AD are selected points K and L, respectively, such that$KA=KD, LA=LB$. Let N - the midpoint of

AC.Prove that if $\angle BNC = \angle DNC$,so $\angle KNL = \angle BCD$

3 There are n balls numbered from 1 to n, and $2n-1$ boxes numbered from 1 to $2n-1$. For each i, ball number i can only be put in the boxes with numbers from 1 to $2i-1$. Let k be an integer from 1 to n. In how many ways we can choose k balls, k boxes and put these balls in the selected boxes so that each box has exactly one ball?

Day 2

1 Let k_1, k_2, k_3 -Excircles triangle $A_1A_2A_3$ with area S. k_1 touch side A_2A_3 at the point B_1 Direct A_1B_1 intersect k_1 at the points B_1 and C_1.Let S_1 - area of the quadrilateral $A_1A_2C_1A_3$ Similarly, we define S_2, S_3. Prove that $\frac{1}{S} \le \frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_2}$

2 Function $f : \mathbb{R} \to \mathbb{R}$ such that $f(xf(y)) = yf(x)$ for any x, y are real numbers. Prove that $f(-x) = -f(x)$ for all real numbers x.

3 The sequence a_n defined as follows: $a_1 = 4, a_2 = 17$ and for any $k \ge 1$ true equalities $a_{2k+1} = a_2 + a_4 + ... + a_{2k} + (k+1)(2^{2k+3} - 1)$ $a_{2k+2} = (2^{2k+2} + 1)a_1 + (2^{2k+3} + 1)a_3 + ... + (2^{3k+1} + 1)a_{2k-1} + k$ Find the smallest m such that $(a_1 + ... a_m)^{2012^{2012}} - 1$ divided $2^{2012^{2012}}$

– Grade level 11

Day 1

1 The number $\overline{13\ldots3}$, with $k > 1$ digits 3, is a prime. Prove that $6 \mid k^2 - 2k + 3$.

2 We call a 6×6 table consisting of zeros and ones *right* if the sum of the numbers in each row and each column is equal to 3. Two right tables are called *similar* if one can get from one to the other by successive interchanges of rows and columns. Find the largest possible size of a set of pairwise similar right tables.

3 Line PQ is tangent to the incircle of triangle ABC in such a way that the points P and Q lie on the sides AB and AC, respectively. On the sides AB and AC are selected points M and N, respectively, so that $AM = BP$ and $AN = CQ$. Prove that all lines constructed in this manner MN pass through one point

Day 2

1 Function $f : \mathbb{R} \to \mathbb{R}$ such that $f(xf(y)) = yf(x)$ for any x, y are real numbers. Prove that $f(-x) = -f(x)$ for all real numbers x.

2 Given the rays OP and OQ.Inside the smaller angle POQ selected points M and N, such that $\angle POM = \angle QON$ and $\angle POM < \angle PON$ The circle, which concern the rays OP and ON, intersects the second circle, which concern the rays OM and OQ at the points B and C. Prove that$\angle POC = \angle QOB$

3 Consider the equation $ax^2 + by^2 = 1$, where a, b are fixed rational numbers. Prove that either such an equation has no solutions in rational numbers, or it has infinitely many solutions.