Kazakhstan National Olympiad 2013

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by hal9v4ik, ts0_9

–	Grade level 9
Day 1	
1	On the board written numbers from 1 to 25 . Bob can pick any three of them say a, b, c and replace by $a^3 + b^3 + c^3$. Prove that last number on the board can not be 2013^3.
2	Prove that for all natural n there exists a, b, c such that $n = \gcd(a, b)(c^2 - ab) + \gcd(b, c)(a^2 - bc) + \gcd(c, a)(b^2 - ca)$.
Day 2	
1	Given triangle ABC with incenter I. Let P,Q be point on circumcircle such that $\angle API = \angle CPI$ and $\angle BQI = \angle CQI$.Prove that BP, AQ and OI are concurrent.
2	a)Does there exist for any rational number $\frac{a}{b}$ some rational numbers $x_1, x_2, ...x_n$ such that $x_1 * x_2 * .... * x_n = 1$ and $x_1 + x_2 + ... + x_n = \frac{a}{b}$ a)Does there exist for any rational number $\frac{a}{b}$ some rational numbers $x_1, x_2, ...x_n$ such that $x_1 * x_2 * .... * x_n = \frac{a}{b}$ and $x_1 + x_2 + ... + x_n = 1$
–	Grade level 10
Day 1	
1	On the board written numbers from 1 to 25 . Bob can pick any three of them say a, b, c and replace by $a^3 + b^3 + c^3$. Prove that last number on the board can not be 2013^3.
2	Let for natural numbers a, b, c and any natural n we have that $(abc)^n$ divides $((a^n - 1)(b^n - 1)(c^n - 1) + 1)^3$. Prove that then $a = b = c$.
3	Let $ABCD$ be cyclic quadrilateral. Let AC and BD intersect at R, and let AB and CD intersect at K. Let M and N are points on AB and CD such that $\frac{AM}{MB} = \frac{CN}{ND}$. Let P and Q be the intersections of MN with the diagonals of $ABCD$. Prove that circumcircles of triangles KMN and PQR are tangent at a fixed point.
Day 2	

1 Find maximum value of $|a^2-bc+1|+|b^2-ac+1|+|c^2-ba+1|$ when a,b,c are reals in $[-2;2]$.

2 Let in triangle ABC incircle touches sides AB,BC,CA at C_1,A_1,B_1 respectively. Let $\frac{2}{CA_1}=\frac{1}{BC_1}+\frac{1}{AC_1}$ .Prove that if X is intersection of incircle and CC_1 then $3CX=CC_1$

3 How many non-intersecting pairs of paths we have from (0,0) to (n,n) so that path can move two ways:top or right?

– Grade level 11

Day 1

1 Find all triples of positive integer (m,n,k) such that $k^m|m^n-1$ and $k^n|n^m-1$

2 Given triangle ABC with incenter I. Let P,Q be point on circumcircle such that $\angle API=\angle CPI$ and $\angle BQI=\angle CQI$.Prove that BP,AQ and OI are concurrent.

3 Consider the following sequence : $a_1=1; a_n=\frac{a_{[\frac{n}{2}]}}{2}+\frac{a_{[\frac{n}{3}]}}{3}+\ldots+\frac{a_{[\frac{n}{n}]}}{n}$. Prove that $a_{2n}<2*a_n (\forall n\in\mathbb{N})$

Day 2

1 Find maximum value of $|a^2-bc+1|+|b^2-ac+1|+|c^2-ba+1|$ when a,b,c are reals in $[-2;2]$.

2 Let in triangle ABC incircle touches sides AB,BC,CA at C_1,A_1,B_1 respectively. Let $\frac{2}{CA_1}=\frac{1}{BC_1}+\frac{1}{AC_1}$ .Prove that if X is intersection of incircle and CC_1 then $3CX=CC_1$

3 How many non-intersecting pairs of paths we have from (0,0) to (n,n) so that path can move two ways:top or right?