Kazakhstan National Olympiad 2014

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Day 1

1 $a_1, a_2, ..., a_{2014}$ is a permutation of $1, 2, 3, ..., 2014$. What is the greatest number of perfect squares can have a set $a_1^2 + a_2, a_2^2 + a_3, a_3^2 + a_4, ..., a_{2013}^2 + a_{2014}, a_{2014}^2 + a_1$?

2 Do there exist positive integers a and b such that $a^n + n^b$ and $b^n + n^a$ are relatively prime for all natural n?

3 The triangle ABC is inscribed in a circle w_1. Inscribed in a triangle circle touchs the sides BC in a point N. w_2 the circle inscribed in a segment BAC circle of w_1, and passing through a point N. Let points O and J the centers of circles w_2 and an extra inscribed circle (touching side BC) respectively. Prove, that lines AO and JN are parallel.

Day 2

1 Given a scalene triangle ABC. Incircle of $\triangle ABC$ touches the sides AB and BC at points C_1 and A_1 respectively, and excircle of $\triangle ABC$ (on side AC) touches AB and BC at points C_2 and A_2 respectively. BN is bisector of $\angle ABC$ (N lies on BC). Lines A_1C_1 and A_2C_2 intersects the line AC at points K_1 and K_2 respectively. Let circumcircles of $\triangle BK_1N$ and $\triangle BK_2N$ intersect circumcircle of a $\triangle ABC$ at points P_1 and P_2 respectively. Prove that AP_1=CP_2

2 $\mathbb{Q}$ is set of all rational numbers. Find all functions $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ such that for all x, y, $z \in \mathbb{Q}$ satisfy $f(x, y) + f(y, z) + f(z, x) = f(0, x + y + z)$

3 Prove that, for all $n \in \mathbb{N}$, on $[n - 4\sqrt{n}, n + 4\sqrt{n}]$ exists natural number $k = x^3 + y^3$ where x, y are nonnegative integers.