Kazakhstan National Olympiad 2015

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Day 1

1 Prove that
$$\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(n+1)^2}<n\cdot\left(1-\frac{1}{\sqrt[n]{2}}\right).$$

2 Solve in positive integers

$x^yy^x=(x+y)^z$

3 A rectangle is said to be $inscribed$ in a triangle if all its vertices lie on the sides of the triangle. Prove that the locus of the centers (the meeting points of the diagonals) of all inscribed in an acute-angled triangle rectangles are three concurrent unclosed segments.

Day 2

4 $P_k(n)$ is the product of all positive divisors of n that are divisible by k (the empty product is equal to 1). Show that $P_1(n)P_2(n)\cdots P_n(n)$ is a perfect square, for any positive integer n.

5 Find all possible $\{x_1,x_2,...x_n\}$ permutations of $\{1,2,...,n\}$ so that when $1\le i\le n-2$ then we have $x_i<x_{i+2}$ and when $1\le i\le n-3$ then we have $x_i<x_{i+3}$. Here $n\ge 4$.

6 The quadrilateral $ABCD$ has an incircle of diameter d which touches BC at K and touches DA at L. Is it always true that the harmonic mean of AB and CD is equal to KL if and only if the geometric mean of AB and CD is equal to d?