Slovenia National Olympiad 2010

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–	First Grade
1	For a real number t and positive real numbers a, b we have $$2a^2 - 3abt + b^2 = 2a^2 + abt - b^2 = 0$$ Find t.
2	Find all prime numbers p, q, r such that $$15p + 7pq + qr = pqr.$$
3	Let ABC be an isosceles triangle with apex at C. Let D and E be two points on the sides AC and BC such that the angle bisectors $\angle DEB$ and $\angle ADE$ meet at F, which lies on segment AB. Prove that F is the midpoint of AB.
4	Find the smallest three-digit number such that the following holds: If the order of digits of this number is reversed and the number obtained by this is added to the original number, the resulting number consists of only odd digits.
5	Let $ABCD$ be a square with the side of 20 units. Amir divides this square into 400 unit squares. Reza then picks 4 of the vertices of these unit squares. These vertices lie inside the square $ABCD$ and define a rectangle with the sides parallel to the sides of the square $ABCD$. There are exactly 24 unit squares which have at least one point in common with the sides of this rectangle. Find all possible values for the area of a rectangle with these properties. *Note:* Vid changed to Amir, and Eva change to Reza!
–	Second Grade
1	Let a, b be real numbers such that $\lvert a\rvert \neq \lvert b\rvert$ and $\frac{a+b}{a-b} + \frac{a-b}{a+b} = 6$. Find the value of the expression $\frac{a^3+b^3}{a^3-b^3} + \frac{a^3-b^3}{a^3+b^3}$.
2	Let a, b and c be nonzero digits. Let p be a prime number which divides the three digit numbers $\overline{abc}$ and $\overline{cba}$. Show that p divides at least one of the numbers $a+b+c, a-b+c$ and $a-c$.

3 Let ABC be an acute triangle. A line parallel to BC intersects the sides AB and AC at D and E, respectively. The circumcircle of the triangle ADE intersects the segment CD at F $(F \neq D)$. Prove that the triangles AFE and CBD are similar.

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4 Let x, y and z be real numbers such that $0 \leq x, y, z \leq 1$. Prove that

$$xyz + (1-x)(1-y)(1-z) \leq 1.$$

When does equality hold?

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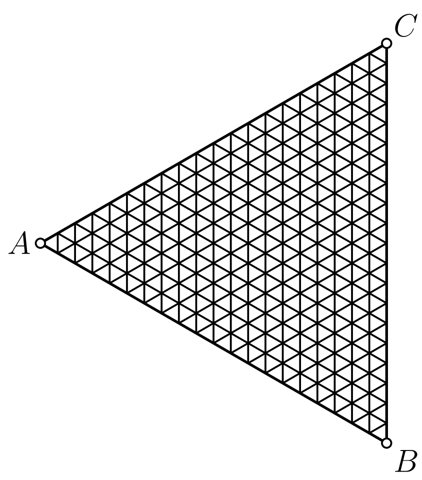
5 Let ABC be an equilateral triangle with the side of 20 units. Amir divides this triangle into 400 smaller equilateral triangles with the sides of 1 unit. Reza then picks 4 of the vertices of these smaller triangles. The vertices lie inside the triangle ABC and form a parallelogram with sides parallel to the sides of the triangle ABC. There are exactly 46 smaller triangles that have at least one point in common with the sides of this parallelogram. Find all possible values for the area of this parallelogram.



[Thanks azjps for drawing the diagram.]

Note: Vid changed to Amir, and Eva change to Reza!

– Third Grade

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1 Let a, b, c be positive integers. Prove that $a^2 + b^2 + c^2$ is divisible by 4, if and only if a, b, c are even.

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2 Find all real x in the interval $[0, 2\pi)$ such that

$$27 \cdot 3^{3\sin x} = 9^{\cos^2 x}.$$

3 Let ABC be an acute triangle with $|AB| > |AC|$. Let D be a point on the side AB, such that the angles $\angle ACD$ and $\angle CBD$ are equal. Let E denote the midpoint of BD, and let S be the circumcenter of the triangle BCD. Prove that the points A, E, S and C lie on the same circle.

4 Find all non-zero real numbers x such that

$$\min\left\{4, x+\frac{4}{x}\right\} \geq 8\min\left\{x, \frac{1}{x}\right\}.$$

5 Ten pirates find a chest filled with golden and silver coins. There are twice as many silver coins in the chest as there are golden. They divide the golden coins in such a way that the difference of the numbers of coins given to any two of the pirates is not divisible by 10. Prove that they cannot divide the silver coins in the same way.

– Fourth Grade

1 Find all prime numbers p, q and r such that $p > q > r$ and the numbers $p-q, p-r$ and $q-r$ are also prime.

2 Let $\mathfrak{K}_1$ and $\mathfrak{K}_2$ be circles centered at O_1 and O_2, respectively, meeting at the points A and B. Let p be the line through the point A meeting the circles $\mathfrak{K}_1$ and $\mathfrak{K}_2$ again at C_1 and C_2. Assume that A lies between C_1 and C_2. Denote the intersection of the lines C_1O_1 and C_2O_2 by D. Prove that the points C_1, C_2, B and D lie on the same circle.

3 Find all functions $f : [0, +\infty) \to [0, +\infty)$ satisfying the equation

$$(y+1)f(x+y) = f\left(xf(y)\right)$$

For all non-negative real numbers x and y.

4 For real numbers a, b and c we have

$$(2b-a)^2 + (2b-c)^2 = 2(2b^2 - ac).$$

Prove that the numbers a, b and c are three consecutive terms in some arithmetic sequence.

5 For what positive integers $n \geq 3$ does there exist a polygon with n vertices (not necessarily convex) with property that each of its sides is parallel to another one of its sides?