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1 A line passing through the incenter I of the triangle ABC intersect its incircle at D and E and its circumcircle at F and G, in such a way that the point D lies between I and F. Prove that: $DF \cdot EG \geq r^2$.

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2 Let $ABCD$ be a tetrahedron and let E, F, G, H, K, L be points lying on the edges AB, BC, CD $, DA, DB, DC$ respectively, in such a way that

$$AE \cdot BE = BF \cdot CF = CG \cdot AG = DH \cdot AH = DK \cdot BK = DL \cdot CL.$$

Prove that the points E, F, G, H, K, L all lie on a sphere.

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3 Let a, b, c be real numbers such that $ab \neq 0$ and $c > 0$. Let $(a_n)_{n \geq 1}$ be the sequence of real numbers defined by: $a_1 = a, a_2 = b$ and

$$a_{n+1} = \frac{a_n^2 + c}{a_{n-1}}$$

for all $n \geq 2$.
Show that all the terms of the sequence are integer numbers if and only if the numbers a, b and $\frac{a^2+b^2+c}{ab}$ are integers.

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4 Let ABC a triangle and P a point such that the triangles PAB, PBC, PCA have the same area and the same perimeter. Prove that if:
a) P is in the interior of the triangle ABC then ABC is equilateral.
b) P is in the exterior of the triangle ABC then ABC is right angled triangle.
