Balkan MO 1995

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1	For all real numbers x, y define $x \star y = \frac{x+y}{1+xy}$. Evaluate the expression $$(\cdots(((2 \star 3) \star 4) \star 5) \star \cdots) \star 1995.$$ *Macedonia*
2	The circles $\mathcal{C}_1(O_1, r_1)$ and $\mathcal{C}_2(O_2, r_2)$, $r_2 > r_1$, intersect at A and B such that $\angle O_1AO_2 = 90^\circ$. The line O_1O_2 meets $\mathcal{C}_1$ at C and D, and $\mathcal{C}_2$ at E and F (in the order C, E, D, F). The line BE meets $\mathcal{C}_1$ at K and AC at M, and the line BD meets $\mathcal{C}_2$ at L and AF at N. Prove that $$\frac{r_2}{r_1} = \frac{KE}{KM} \cdot \frac{LN}{LD}.$$ *Greece*
3	Let a and b be natural numbers with $a > b$ and having the same parity. Prove that the solutions of the equation $$x^2 - (a^2 - a + 1)(x - b^2 - 1) - (b^2 + 1)^2 = 0$$ are natural numbers, none of which is a perfect square. *Albania*
4	Let n be a positive integer and $\mathcal{S}$ be the set of points (x, y) with $x, y \in \{1, 2, \ldots, n\}$. Let $\mathcal{T}$ be the set of all squares with vertices in the set $\mathcal{S}$. We denote by a_k ($k \geq 0$) the number of (unordered) pairs of points for which there are exactly k squares in $\mathcal{T}$ having these two points as vertices. Prove that $a_0 = a_2 + 2a_3$. *Yugoslavia*