Balkan MO 1997

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1	Suppose that O is a point inside a convex quadrilateral $ABCD$ such that $$OA^2+OB^2+OC^2+OD^2=2\mathcal{A}[ABCD],$$ where by $\mathcal{A}[ABCD]$ we have denoted the area of $ABCD$. Prove that $ABCD$ is a square and O is its center. *Yugoslavia*
2	Let $S=\{A_1,A_2,\ldots,A_k\}$ be a collection of subsets of an n-element set A. If for any two elements $x,y\in A$ there is a subset $A_i\in S$ containing exactly one of the two elements x, y, prove that $2^k\geq n$. *Yugoslavia*
3	The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ touch each other externally at D, and touch a circle ω internally at B and C, respectively. Let A be an intersection point of ω and the common tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ at D. Lines AB and AC meet $\mathcal{C}_1$ and $\mathcal{C}_2$ again at K and L, respectively, and the line BC meets $\mathcal{C}_1$ again at M and $\mathcal{C}_2$ again at N. Prove that the lines AD, KM, LN are concurrent. *Greece*
4	Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $$f(xf(x)+f(y))=f^2(x)+y$$ for all $x,y\in\mathbb{R}$.