Balkan MO 2001

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1 Let a, b, n be positive integers such that $2^n - 1 = ab$. Let $k \in \mathbb{N}$ such that $ab + a - b - 1 \equiv 0 \pmod{2^k}$ and $ab + a - b - 1 \neq 0 \pmod{2^{k+1}}$. Prove that k is even.

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2 A convex pentagon $ABCDE$ has rational sides and equal angles. Show that it is regular.

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3 Let a, b, c be positive real numbers with $abc \leq a + b + c$. Show that

$$a^2 + b^2 + c^2 \geq \sqrt{3}abc.$$

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4 A cube side 3 is divided into 27 unit cubes. The unit cubes are arbitrarily labeled 1 to 27 (each cube is given a different number). A move consists of swapping the cube labeled 27 with one of its 6 neighbours. Is it possible to find a finite sequence of moves at the end of which cube 27 is in its original position, but cube n has moved to the position originally occupied by $27 - n$ (for each $n = 1, 2, \ldots, 26$)?
