Balkan MO 2002

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1 Consider n points $A_1, A_2, A_3, \dots, A_n$ ($n \geq 4$) in the plane, such that any three are not collinear. Some pairs of distinct points among $A_1, A_2, A_3, \dots, A_n$ are connected by segments, such that every point is connected with at least three different points. Prove that there exists $k > 1$ and the distinct points $X_1, X_2, \dots, X_{2k}$ in the set $\{A_1, A_2, A_3, \dots, A_n\}$, such that for every $i \in \overline{1, 2k-1}$ the point X_i is connected with X_{i+1}, and X_{2k} is connected with X_1.

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2 Let the sequence $\{a_n\}_{n\geq 1}$ be defined by $a_1 = 20$, $a_2 = 30$ and $a_{n+2} = 3a_{n+1} - a_n$ for all $n \geq 1$. Find all positive integers n such that $1 + 5a_n a_{n+1}$ is a perfect square.

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3 Two circles with different radii intersect in two points A and B. Let the common tangents of the two circles be MN and ST such that M, S lie on the first circle, and N, T on the second. Prove that the orthocenters of the triangles AMN, AST, BMN and BST are the four vertices of a rectangle.

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4 Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for every positive integer n we have:

$$2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2002.$$
