Balkan MO 2007

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1 Let $ABCD$ a convex quadrilateral with $AB = BC = CD$, with AC not equal to BD and E be the intersection point of it's diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120$.

2 Find all real functions f defined on $\mathbb{R}$, such that

$$f(f(x) + y) = f(f(x) - y) + 4f(x)y,$$

for all real numbers x, y.

3 Find all positive integers n such that there exist a permutation σ on the set $\{1, 2, 3, \ldots, n\}$ for which

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\ldots + \sqrt{\sigma(n-1) + \sqrt{\sigma(n)}}}}}$$

is a rational number.

4 For a given positive integer $n > 2$, let C_1, C_2, C_3 be the boundaries of three convex $n-$ gons in the plane , such that
$C_1 \cap C_2, C_2 \cap C_3, C_1 \cap C_3$ are finite. Find the maximum number of points of the sets $C_1 \cap C_2 \cap C_3$.