Balkan MO 2010

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1	Let a, b and c be positive real numbers. Prove that $$\frac{a^2b(b-c)}{a+b}+\frac{b^2c(c-a)}{b+c}+\frac{c^2a(a-b)}{c+a}\geq 0.$$
2	Let ABC be an acute triangle with orthocentre H, and let M be the midpoint of AC. The point C_1 on AB is such that CC_1 is an altitude of the triangle ABC. Let H_1 be the reflection of H in AB. The orthogonal projections of C_1 onto the lines AH_1, AC and BC are P, Q and R, respectively. Let M_1 be the point such that the circumcentre of triangle PQR is the midpoint of the segment MM_1. Prove that M_1 lies on the segment BH_1.
3	A strip of width w is the set of all points which lie on, or between, two parallel lines distance w apart. Let S be a set of n $(n\geq 3)$ points on the plane such that any three different points of S can be covered by a strip of width 1. Prove that S can be covered by a strip of width 2.
4	For each integer n $(n\geq 2)$, let $f(n)$ denote the sum of all positive integers that are at most n and not relatively prime to n. Prove that $f(n+p)\neq f(n)$ for each such n and every prime p.