Ukraine National Mathematical Olympiad 2009

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– Grade level 8

Day 1

1 Find all positive integer solutions of equation $n^3 - 2 = k!$.

2 There is convex 2009-gon on the plane.

a) Find the greatest number of vertices of 2009-gon such that no two forms the side of the polygon.

b) Find the greatest number of vertices of 2009-gon such that among any three of them there is one that is not connected with other two by side.

3 On the party every boy gave 1 candy to every girl and every girl gave 1 candy to every boy. Then every boy ate 2 candies and every girl ate 3 candies. It is known that $\frac{1}{4}$ of all candies was eaten. Find the greatest possible number of children on the party.

4 In the triangle ABC given that $\angle ABC = 120^\circ$. The bisector of $\angle B$ meet AC at M and external bisector of $\angle BCA$ meet AB at P. Segments MP and BC intersects at K. Prove that $\angle AKM = \angle KPC$.

Day 2

1 Let a, b, c be integers satisfying $ab + bc + ca = 1$. Prove that $(1 + a^2)(1 + b^2)(1 + c^2)$ is a perfect square.

2 In acute-angled triangle ABC, let M be the midpoint of BC and let K be a point on side AB. We know that AM meet CK at F. Prove that if $AK = KF$ then $AB = CF$.

3 Given 2009×4018 rectangular board. Frame is a rectangle $n \times n$ or $n \times (n + 2)$ for $(n \geq 3)$ without all cells which dont have any common points with boundary of rectangle. Rectangles $1 \times 1, 1 \times 2, 1 \times 3$ and 2×4 are also frames. Two players by turn paint all cells of some frame that has no painted cells yet. Player that can't make such move loses. Who has a winning strategy?

4 **)** Prove that for any positive integer n there exist a pair of positive integers (m, k) such that

$$k + m^k + n^{m^k} = 2009^n.$$

b) Prove that there are infinitely many positive integers n for which there is only one such pair.

– Grade level 9

Day 1

1 Build the set of points (x, y) on coordinate plane, that satisfies equality:

$$\sqrt{1-x^2}+\sqrt{1-y^2}=2-x^2-y^2.$$

2 On the party every boy gave 1 candy to every girl and every girl gave 1 candy to every boy. Then every boy ate 2 candies and every girl ate 3 candies. It is known that $\frac{1}{4}$ of all candies was eaten. Find the greatest possible number of children on the party.

3 In triangle ABC points M, N are midpoints of BC, CA respectively. Point P is inside ABC such that $\angle BAP = \angle PCA = \angle MAC$. Prove that $\angle PNA = \angle AMB$.

4 Let $x \leq y \leq z \leq t$ be real numbers such that $xy + xz + xt + yz + yt + zt = 1$.

a) Prove that $xt < \frac{1}{3}$,

b) Find the least constant C for which inequality $xt < C$ holds for all possible values x and t.

Day 2

1 Pairwise distinct real numbers a, b, c satisfies the equality

$$a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}.$$

Find all possible values of abc.

2 Find all prime numbers p and positive integers m such that $2p^2 + p + 9 = m^2$.

3 Given 2009×4018 rectangular board. Frame is a rectangle $n \times n$ or $n \times (n + 2)$ for $(n \geq 3)$ without all cells which dont have any common points with boundary of rectangle. Rectangles $1\times1, 1\times2, 1\times3$ and 2×4 are also frames. Two players by turn paint all cells of some frame that has no painted cells yet. Player that can't make such move loses. Who has a winning strategy?

4 In the trapezoid $ABCD$ we know that $CD \perp BC$, and $CD \perp AD$. Circle w with diameter AB intersects AD in points A and P, tangent from P to w intersects CD at M. The second tangent from M to w touches w at Q. Prove that midpoint of CD lies on BQ.

– Grade level 10

Day 1

1 Compare the number of distinct prime divisors of $200^2 \cdot 201^2 \cdot ... \cdot 900^2$ and $(200^2-1)(201^2-1) \cdot ... \cdot (900^2-1)$.

2 There is a knight in the left down corner of 2009×2009 chessboard. The row and the column containing this corner are painted. The knight cannot move into painted cell and after its move new row and column that contains a square with knight become painted. Is it possible to paint all rows and columns of the chessboard?

3 In triangle ABC points M, N are midpoints of BC, CA respectively. Point P is inside ABC such that $\angle BAP = \angle PCA = \angle MAC$. Prove that $\angle PNA = \angle AMB$.

4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x + xy + f(y)\right) = \left(f(x) + \frac{1}{2}\right)\left(f(y) + \frac{1}{2}\right) \qquad \forall x, y \in \mathbb{R}.$$

Day 2

1 Pairwise distinct real numbers a, b, c satisfies the equality

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}.$$

Find all possible values of abc.

2 Find all prime numbers p and positive integers m such that $2p^2 + p + 9 = m^2$.

3 Given a $n \times n$ square board. Two players by turn remove some side of unit square if this side is not a bound of $n \times n$ square board. The player lose if after his move $n \times n$ square board became broken into two parts. Who has a winning strategy?

4 Let $ABCD$ be a parallelogram with $\angle BAC = 45^\circ$, and $AC > BD$. Let w_1 and w_2 be two circles with diameters AC and DC, respectively. The circle w_1 intersects AB at E and the circle w_2 intersects AC at O and C, and AD at F. Find the ratio of areas of triangles AOE and COF if $AO = a$, and $FO = b$.

– Grade level 11

Day 1

1 Find all possible real values of a for which the system of equations

$$\{\begin{array}{c} x+y+z=0\\ xy+yz+azx=0 \end{array}$$

has exactly one solution.

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2 Let $M=\{1,2,3,4,6,8,12,16,24,48\}$. Find out which of four-element subsets of M are more: those with product of all elements greater than 2009 or those with product of all elements less than 2009.

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3 In triangle ABC let M and N be midpoints of BC and AC, respectively. Point P is inside ABC such that $\angle BAP=\angle PBC=\angle PCA$. Prove that if $\angle PNA=\angle AMB$, then ABC is isosceles triangle.

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4 Find all polynomials $P(x)$ with real coefficients such that for all pairwise distinct positive integers x,y,z,t with $x^2+y^2+z^2=2t^2$ and $\gcd(x,y,z,t)=1$, the following equality holds

$$2P^2(t)+2P(xy+yz+zx)=P^2(x+y+z).$$

Note. $P^2(k)=\left(P(k)\right)^2$.

Day 2

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1 Solve the system of equations

$$\{\begin{array}{c} x^3=2y^3+y-2\\ y^3=2z^3+z-2\\ z^3=2x^3+x-2 \end{array}$$

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2 Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that

$$f(n|m|)+f(n(|m|+2))=2f(n(|m|+1)) \qquad \forall m,n\in\mathbb{Z}.$$

Note. $|x|$ denotes the absolute value of the integer x.

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3 Point O is inside triangle ABC such that $\angle AOB=\angle BOC=\angle COA=120^\circ$. Prove that

$$\frac{AO^2}{BC}+\frac{BO^2}{CA}+\frac{CO^2}{AB}\geq\frac{AO+BO+CO}{\sqrt{3}}.$$

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4 Let G be a connected graph, with degree of all vertices not less then $m \geq 3$, such that there is no path through all vertices of G being in every vertex exactly once. Find the least possible number of vertices of G.
