NIMO Problems 2012

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by AIME15, v_Enhance

–	Winter Contest
–	January 22nd
1	In a 10 by 10 grid of dots, what is the maximum number of lines that can be drawn connecting two dots on the grid so that no two lines are parallel? *Proposed by Aaron Lin*
2	If r_1, r_2, and r_3 are the solutions to the equation $x^3 - 5x^2 + 6x - 1 = 0$, then what is the value of $r_1^2 + r_2^2 + r_3^2$? *Proposed by Eugene Chen*
3	The expression $\circ 1 \ \circ \ 2 \ \circ 3 \ \circ \cdots \circ \ 2012$ is written on a blackboard. Catherine places a $+$ sign or a $-$ sign into each blank. She then evaluates the expression, and finds the remainder when it is divided by 2012. How many possible values are there for this remainder? *Proposed by Aaron Lin*
4	Parallel lines ℓ_1 and ℓ_2 are drawn in a plane. Points $A_1, A_2, \dots, A_n$ are chosen on ℓ_1, and points $B_1, B_2, \dots, B_{n+1}$ are chosen on ℓ_2. All segments A_iB_j are drawn, such that $1 \le i \le n$ and $1 \le j \le n+1$. Let the number of total intersections between these segments (not including endpoints) be denoted by Q. Given that no three segments are concurrent, besides at endpoints, prove that Q is divisible by 3. *Proposed by Lewis Chen*
5	In convex hexagon $ABCDEF$, $\angle A \cong \angle B$, $\angle C \cong \angle D$, and $\angle E \cong \angle F$. Prove that the perpendicular bisectors of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ pass through a common point. *Proposed by Lewis Chen*
6	The positive numbers a, b, c satisfy $4abc(a+b+c) = (a+b)^2(a+c)^2$. Prove that $a(a+b+c) = bc$. *Proposed by Aaron Lin*
7	For how many positive integers $n \le 500$ is $n!$ divisible by 2^{n-2}? *Proposed by Eugene Chen*

8 A convex 2012-gon $A_1A_2A_3\dots A_{2012}$ has the property that for every integer $1 \le i \le 1006$, $\overline{A_iA_{i+1006}}$ partitions the polygon into two congruent regions. Show that for every pair of integers $1 \le j < k \le 1006$, quadrilateral $A_jA_kA_{j+1006}A_{k+1006}$ is a parallelogram.

Proposed by Lewis Chen

– Monthly Contests

Day 1 September 17th

1 Dan the dog spots Cate the cat 50m away. At that instant, Cate begins running away from Dan at 6 m/s, and Dan begins running toward Cate at 8 m/s. Both of them accelerate instantaneously and run in straight lines. Compute the number of seconds it takes for Dan to reach Cate.

Proposed by Eugene Chen

2 A permutation $(a_1, a_2, a_3, \dots, a_{100})$ of $(1, 2, 3, \dots, 100)$ is chosen at random. Denote by p the probability that $a_{2i} > a_{2i-1}$ for all $i \in \{1, 2, 3, \dots, 50\}$. Compute the number of ordered pairs of positive integers (a, b) satisfying $\frac{1}{a^b} = p$.

Proposed by Aaron Lin

3 For positive integers $1 \le n \le 100$, let

$$f(n) = \sum_{i=1}^{100} i\left|i-n\right|.$$

Compute $f(54) - f(55)$.

Proposed by Aaron Lin

4 In $\triangle ABC$, $AB = AC$. Its circumcircle, Γ, has a radius of 2. Circle Ω has a radius of 1 and is tangent to Γ, $\overline{AB}$, and $\overline{AC}$. The area of $\triangle ABC$ can be expressed as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c, where b is squarefree and $\gcd(a, c) = 1$. Compute $a + b + c$.

Proposed by Aaron Lin

5 If $w = a + bi$, where a and b are real numbers, then $\Re(w) = a$ and $\Im(w) = b$. Let $z = c + di$, where $c, d \ge 0$. If

$$\Re(z) + \Im(z) = 7,$$
$$\Re(z^2) + \Im(z^2) = 17,$$

then compute $\left|\Re\left(z^3\right) + \Im\left(z^3\right)\right|$.

Proposed by Lewis Chen

6 A square is called *proper* if its sides are parallel to the coordinate axes. Point P is randomly selected inside a proper square S with side length 2012. Denote by T the largest proper square that lies within S and has P on its perimeter, and denote by a the expected value of the side length of T. Compute $\lfloor a \rfloor$, the greatest integer less than or equal to a.

Proposed by Lewis Chen

7 Point P lies in the interior of rectangle $ABCD$ such that $AP + CP = 27$, $BP - DP = 17$, and $\angle DAP \cong \angle DCP$. Compute the area of rectangle $ABCD$.

Proposed by Aaron Lin

8 The positive integer-valued function $f(n)$ satisfies $f(f(n)) = 4n$ and $f(n+1) > f(n) > 0$ for all positive integers n. Compute the number of possible 16-tuples $(f(1), f(2), f(3), \dots, f(16))$.

Proposed by Lewis Chen

Day 2 October 17th

1 Compute the largest integer $N \le 2012$ with four distinct digits.

Proposed by Evan Chen

2 A *normal magic square* of order n is an arrangement of the integers from 1 to n^2 in a square such that the n numbers in each row, each column, and each of the two diagonals sum to a constant, called the *magic sum* of the magic square. Compute the magic sum of a normal magic square of order 8.

3 A polygon $A_1A_2A_3\dots A_n$ is called *beautiful* if there exist indices i, j, and k such that $\measuredangle A_iA_jA_k = 144^\circ$. Compute the number of integers $3 \le n \le 2012$ for which a regular n-gon is beautiful.

Proposed by Aaron Lin

4 When flipped, coin A shows heads $\frac{1}{3}$ of the time, coin B shows heads $\frac{1}{2}$ of the time, and coin C shows heads $\frac{2}{3}$ of the time. Anna selects one of the coins at random and flips it four times, yielding three heads and one tail. The probability that Anna flipped coin A can be expressed as $\frac{p}{q}$ for relatively prime positive integers p and q. Compute $p + q$.

Proposed by Eugene Chen

5 In $\triangle ABC$, $AB = 30$, $BC = 40$, and $CA = 50$. Squares A_1A_2BC, B_1B_2AC, and C_1C_2AB are erected outside $\triangle ABC$, and the pairwise intersections of lines A_1A_2, B_1B_2, and C_1C_2 are P, Q, and R. Compute the length of the shortest altitude of $\triangle PQR$.

Proposed by Lewis Chen

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6 In $\triangle ABC$ with circumcenter O, $\measuredangle A = 45^\circ$. Denote by X the second intersection of $\overrightarrow{AO}$ with the circumcircle of $\triangle BOC$. Compute the area of quadrilateral $ABXC$ if $BX = 8$ and $CX = 15$.

Proposed by Aaron Lin

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7 The sequence $\{a_i\}_{i\ge 1}$ is defined by $a_1 = 1$ and

$$a_n = \lfloor a_{n-1} + \sqrt{a_{n-1}} \rfloor$$

for all $n \ge 2$. Compute the eighth perfect square in the sequence.

Proposed by Lewis Chen

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8 Compute the number of sequences of real numbers $a_1, a_2, a_3, \dots, a_{16}$ satisfying the condition that for every positive integer n,

$$a_1^n + a_2^{2n} + \cdots + a_{16}^{16n} = \begin{cases} 10^{n+1} + 10^n + 1 & \text{for even } n \\ 10^n - 1 & \text{for odd } n \end{cases}.$$

Proposed by Evan Chen

Day 3 November 24th

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1 Hexagon $ABCDEF$ is inscribed in a circle. If $\measuredangle ACE = 35^\circ$ and $\measuredangle CEA = 55^\circ$, then compute the sum of the degree measures of $\angle ABC$ and $\angle EFA$.

Proposed by Isabella Grabski

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2 Compute the number of positive integers $n < 2012$ that share exactly two positive factors with 2012.

Proposed by Aaron Lin

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3 Compute the sum of the distinct prime factors of 10101.

Proposed by Lewis Chen

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4 The *subnumbers* of an integer n are the numbers that can be formed by using a contiguous subsequence of the digits. For example, the subnumbers of 135 are 1, 3, 5, 13, 35, and 135. Compute the number of primes less than 1,000,000,000 that have no non-prime subnumbers. One such number is 37, because 3, 7, and 37 are prime, but 135 is not one, because the subnumbers 1, 35, and 135 are not prime.

Proposed by Lewis Chen

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5 The hour and minute hands on a certain 12-hour analog clock are indistinguishable. If the hands of the clock move continuously, compute the number of times strictly between noon and midnight for which the information on the clock is not sufficient to determine the time.

Proposed by Lewis Chen

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6 In rhombus $NIMO$, $MN = 150\sqrt{3}$ and $\measuredangle MON = 60^\circ$. Denote by S the locus of points P in the interior of $NIMO$ such that $\angle MPO \cong \angle NPO$. Find the greatest integer not exceeding the perimeter of S.

Proposed by Evan Chen

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7 For every pair of reals $0 < a < b < 1$, we define sequences $\{x_n\}_{n\ge 0}$ and $\{y_n\}_{n\ge 0}$ by $x_0 = 0$, $y_0 = 1$, and for each integer $n \ge 1$:

$$x_n = (1-a)x_{n-1} + ay_{n-1},$$
$$y_n = (1-b)x_{n-1} + by_{n-1}.$$

The *supermean* of a and b is the limit of $\{x_n\}$ as n approaches infinity. Over all pairs of real numbers (p, q) satisfying $\left(p - \frac{1}{2}\right)^2 + \left(q - \frac{1}{2}\right)^2 \le \left(\frac{1}{10}\right)^2$, the minimum possible value of the supermean of p and q can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

Proposed by Lewis Chen

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8 Concentric circles Ω_1 and Ω_2 with radii 1 and 100, respectively, are drawn with center O. Points A and B are chosen independently at random on the circumferences of Ω_1 and Ω_2, respectively. Denote by ℓ the tangent line to Ω_1 passing through A, and denote by P the reflection of B across ℓ. Compute the expected value of OP^2.

Proposed by Lewis Chen

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9 Let $f(x) = x^2 - 2x$. A set of real numbers S is *valid* if it satisfies the following: • If $x \in S$, then $f(x) \in S$. • If $x \in S$ and $\underbrace{f(f(\dots f}_{k\ f\text{'s}}(x)\dots)) = x$ for some integer k, then $f(x) = x$.

Compute the number of 7-element valid sets.

Proposed by Lewis Chen

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10 For reals $x_1, x_2, x_3, \dots, x_{333} \in [-1, \infty)$, let $S_k = \sum_{i=1}^{333} x_i^k$ for each k. If $S_2 = 777$, compute the least possible value of S_3.

Proposed by Evan Chen

Day 4 December 17th

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1 Compute the average of the integers $2, 3, 4, \ldots, 2012$.

Proposed by Eugene Chen

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2 For which positive integer n is the quantity $\frac{n}{3} + \frac{40}{n}$ minimized?

Proposed by Eugene Chen

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3 In chess, there are two types of minor pieces, the bishop and the knight. A bishop may move along a diagonal, as long as there are no pieces obstructing its path. A knight may jump to any lattice square $\sqrt{5}$ away as long as it isn't occupied.

One day, a bishop and a knight were on squares in the same row of an infinite chessboard, when a huge meteor storm occurred, placing a meteor in each square on the chessboard independently and randomly with probability p. Neither the bishop nor the knight were hit, but their movement may have been obstructed by the meteors.

The value of p that would make the expected number of valid squares that the bishop can move to and the number of squares that the knight can move to equal can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

Proposed by Lewis Chen

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4 Let $S = \{(x, y) : x, y \in \{1, 2, 3, \ldots, 2012\}\}$. For all points (a, b), let $N(a, b) = \{(a-1, b), (a+1, b), (a, b-1), (a, b+1)\}$. Kathy constructs a set T by adding n distinct points from S to T at random. If the expected value of $\displaystyle\sum_{(a,b)\in T} |N(a, b) \cap T|$ is 4, then compute n.

Proposed by Lewis Chen

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5 A number is called *purple* if it can be expressed in the form $\frac{1}{2^a 5^b}$ for positive integers $a > b$. The sum of all purple numbers can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

Proposed by Eugene Chen

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6 The polynomial $P(x) = x^3 + \sqrt{6}x^2 - \sqrt{2}x - \sqrt{3}$ has three distinct real roots. Compute the sum of all $0 \le \theta < 360$ such that $P(\tan\theta^\circ) = 0$.

Proposed by Lewis Chen

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7 In quadrilateral $ABCD$, $AC = BD$ and $\measuredangle B = 60^\circ$. Denote by M and N the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. If $MN = 12$ and the area of quadrilateral $ABCD$ is 420, then compute AC.

Proposed by Aaron Lin

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8 Bob has invented the Very Normal Coin (VNC). When the VNC is flipped, it shows heads $\frac{1}{2}$ of the time and tails $\frac{1}{2}$ of the time - unless it has yielded the same result five times in a row, in which case it is guaranteed to yield the opposite result. For example, if Bob flips five heads in a row, then the next flip is guaranteed to be tails.

Bob flips the VNC an infinite number of times. On the nth flip, Bob bets 2^{-n} dollars that the VNC will show heads (so if the second flip shows heads, Bob wins $\$0.25$, and if the third flip shows tails, Bob loses $\$0.125$).

Assume that dollars are infinitely divisible. Given that the first flip is heads, the expected number of dollars Bob is expected to win can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

Proposed by Lewis Chen

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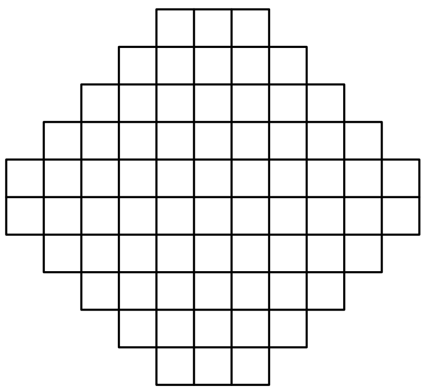
9 In how many ways can the following figure be tiled with 2×1 dominos?



Proposed by Lewis Chen

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10 In cyclic quadrilateral $ABXC$, $\measuredangle XAB = \measuredangle XAC$. Denote by I the incenter of $\triangle ABC$ and by D the projection of I on $\overline{BC}$. If $AI = 25$, $ID = 7$, and $BC = 14$, then XI can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

Proposed by Aaron Lin

– Summer Contest

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1 Let $f(x) = (x^4 + 2x^3 + 4x^2 + 2x + 1)^5$. Compute the prime p satisfying $f(p) = 418{,}195{,}493$.

Proposed by Eugene Chen

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2 Compute the number of positive integers n satisfying the inequalities

$$2^{n-1} < 5^{n-3} < 3^n.$$

Proposed by Isabella Grabski

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3 Let

$$S = \sum_{i=1}^{2012} i!.$$

The tens and units digits of S (in decimal notation) are a and b, respectively. Compute $10a + b$.

Proposed by Lewis Chen

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4 The degree measures of the angles of nondegenerate hexagon $ABCDEF$ are integers that form a non-constant arithmetic sequence in some order, and $\angle A$ is the smallest angle of the (not necessarily convex) hexagon. Compute the sum of all possible degree measures of $\angle A$.

Proposed by Lewis Chen

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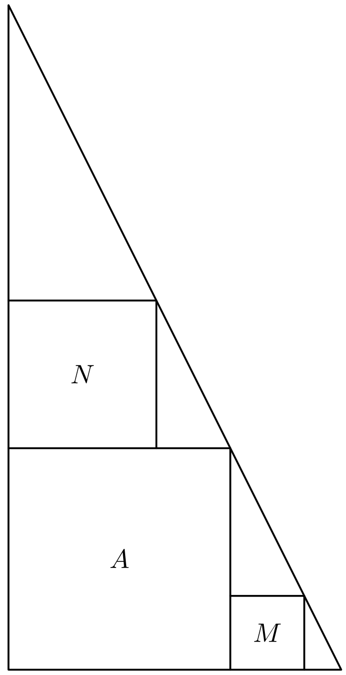
5 In the diagram below, three squares are inscribed in right triangles. Their areas are A, M, and N, as indicated in the diagram. If $M = 5$ and $N = 12$, then A can be expressed as $a + b\sqrt{c}$, where a, b, and c are positive integers and c is not divisible by the square of any prime. Compute $a + b + c$.



Proposed by Aaron Lin

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6 When Eva counts, she skips all numbers containing a digit divisible by 3. For example, the first ten numbers she counts are 1, 2, 4, 5, 7, 8, 11, 12, 14, 15. What is the 100^{th} number she counts?

Proposed by Eugene Chen

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7 A permutation $(a_1, a_2, a_3, \ldots, a_{2012})$ of $(1, 2, 3, \ldots, 2012)$ is selected at random. If S is the expected value of

$$\sum_{i=1}^{2012} |a_i - i|,$$

then compute the sum of the prime factors of S.

Proposed by Aaron Lin

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8 Points A, B, and O lie in the plane such that $\measuredangle AOB = 120^\circ$. Circle ω_0 with radius 6 is constructed tangent to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$. For all $i \ge 1$, circle ω_i with radius r_i is constructed such that $r_i < r_{i-1}$ and ω_i is tangent to $\overrightarrow{OA}$, $\overrightarrow{OB}$, and ω_{i-1}. If

$$S = \sum_{i=1}^{\infty} r_i,$$

then S can be expressed as $a\sqrt{b}+c$, where a, b, c are integers and b is not divisible by the square of any prime. Compute $100a + 10b + c$.

Proposed by Aaron Lin

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9 A quadratic polynomial $p(x)$ with integer coefficients satisfies $p(41) = 42$. For some integers $a, b > 41$, $p(a) = 13$ and $p(b) = 73$. Compute the value of $p(1)$.

Proposed by Aaron Lin

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10 A *triangulation* of a polygon is a subdivision of the polygon into triangles meeting edge to edge, with the property that the set of triangle vertices coincides with the set of vertices of the polygon. Adam randomly selects a triangulation of a regular 180-gon. Then, Bob selects one of the 178 triangles in this triangulation. The expected number of 1° angles in this triangle can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

Proposed by Lewis Chen

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11 Let a and b be two positive integers satisfying the equation

$$20\sqrt{12} = a\sqrt{b}.$$

Compute the sum of all possible distinct products ab.

Proposed by Lewis Chen

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12 The NEMO (National Electronic Math Olympiad) is similar to the NIMO Summer Contest, in that there are fifteen problems, each worth a set number of points. However, the NEMO is weighted using Fibonacci numbers; that is, the n^{th} problem is worth F_n points, where $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. The two problem writers are fair people, so they make sure that each of them is responsible for problems worth an equal number of total points. Compute the number of ways problem writing assignments can be distributed between the two writers.

Proposed by Lewis Chen

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13 For the NEMO, Kevin needs to compute the product

$$9 \times 99 \times 999 \times \cdots \times 999999999.$$

Kevin takes exactly ab seconds to multiply an a-digit integer by a b-digit integer. Compute the minimum number of seconds necessary for Kevin to evaluate the expression together by performing eight such multiplications.

Proposed by Evan Chen

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14 A set of lattice points is called *good* if it does not contain two points that form a line with slope -1 or slope 1. Let $S = \{(x, y) \mid x, y \in \mathbb{Z}, 1 \leq x, y \leq 4\}$. Compute the number of non-empty good subsets of S.

Proposed by Lewis Chen

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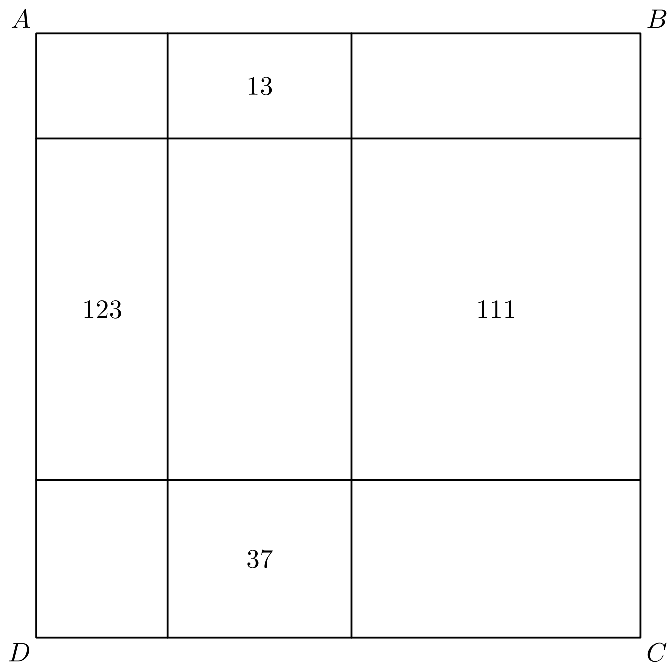
15 In the diagram below, square $ABCD$ with side length 23 is cut into nine rectangles by two lines parallel to $\overline{AB}$ and two lines parallel to $\overline{BC}$. The areas of four of these rectangles are indicated in the diagram. Compute the largest possible value for the area of the central rectangle.



Proposed by Lewis Chen
