NIMO Problems 2013

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by v_Enhance, ahaanomegas

–	Monthly Contests
Day 5	January 24th

1 Tim is participating in the following three math contests. On each contest his score is the number of correct answers.

- The Local Area Inspirational Math Exam consists of 15 problems.
- The Further Away Regional Math League has 10 problems.
- The Distance-Optimized Math Open has 50 problems.

For every positive integer n, Tim knows the answer to the nth problems on each contest (which are pairwise distinct), if they exist; however, these answers have been randomly permuted so that he does not know which answer corresponds to which contest. Unaware of the shuffling, he competes with his modified answers. Compute the expected value of the sum of his scores on all three contests.

Proposed by Evan Chen

2 The cost of five water bottles is \$13, rounded to the nearest dollar, and the cost of six water bottles is \$16, also rounded to the nearest dollar. If all water bottles cost the same integer number of cents, compute the number of possible values for the cost of a water bottle.

Proposed by Eugene Chen

3 In triangle ABC, $AB = 13$, $BC = 14$ and $CA = 15$. Segment BC is split into $n + 1$ congruent segments by n points. Among these points are the feet of the altitude, median, and angle bisector from A. Find the smallest possible value of n.

Proposed by Evan Chen

4 The infinite geometric series of positive reals $a_1, a_2, \ldots$ satisfies

$$1 = \sum_{n=1}^{\infty} a_n = -\frac{1}{2013} + \sum_{n=1}^{\infty} \text{GM}(a_1, a_2, \ldots, a_n) = \frac{1}{N} + a_1$$

where $\text{GM}(x_1, x_2, \ldots, x_k) = \sqrt[k]{x_1 x_2 \cdots x_k}$ denotes the geometric mean. Compute N.

Proposed by Aaron Lin

5 Compute the number of five-digit positive integers $\overline{vwxyz}$ for which

$$(10v + w) + (10w + x) + (10x + y) + (10y + z) = 100.$$

Proposed by Evan Chen

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6 Tom has a scientific calculator. Unfortunately, all keys are broken except for one row: 1, 2, 3, + and -.
Tom presses a sequence of 5 random keystrokes; at each stroke, each key is equally likely to be pressed. The calculator then evaluates the entire expression, yielding a result of E. Find the expected value of E.
(Note: Negative numbers are permitted, so 13-22 gives $E = -9$. Any excess operators are parsed as signs, so -2-+3 gives $E = -5$ and -+-31 gives $E = 31$. Trailing operators are discarded, so 2++-+ gives $E = 2$. A string consisting only of operators, such as -++-+, gives $E = 0$.)

Proposed by Lewis Chen

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7 For each integer $k \geq 2$, the decimal expansions of the numbers $1024, 1024^2, \dots, 1024^k$ are concatenated, in that order, to obtain a number X_k. (For example, $X_2 = 10241048576$.) If

$$\frac{X_n}{1024^n}$$

is an odd integer, find the smallest possible value of n, where $n \geq 2$ is an integer.

Proposed by Evan Chen

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8 Let $AXYZB$ be a convex pentagon inscribed in a semicircle with diameter AB. Suppose that $AZ - AX = 6$, $BX - BZ = 9$, $AY = 12$, and $BY = 5$. Find the greatest integer not exceeding the perimeter of quadrilateral $OXYZ$, where O is the midpoint of AB.

Proposed by Evan Chen

Day 6 February 24th

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1 Find the sum of all primes that can be written both as a sum of two primes and as a difference of two primes.

Anonymous Proposal

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2 Let f be a function from positive integers to positive integers where $f(n) = \frac{n}{2}$ if n is even and $f(n) = 3n + 1$ if n is odd. If a is the smallest positive integer satisfying

$$\underbrace{f(f(\cdots f}_{2013\ f\text{'s}}(a)\cdots)) = 2013,$$

find the remainder when a is divided by 1000.

Based on a proposal by Ivan Koswara

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3 Find the integer $n \geq 48$ for which the number of trailing zeros in the decimal representation of $n!$ is exactly $n - 48$.

Proposed by Kevin Sun

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4 While taking the SAT, you become distracted by your own answer sheet. Because you are not bound to the College Board's limiting rules, you realize that there are actually 32 ways to mark your answer for each question, because you could fight the system and bubble in multiple letters at once: for example, you could mark AB, or AC, or ABD, or even $ABCDE$, or nothing at all!

You begin to wonder how many ways you could mark off the 10 questions you haven't yet answered. To increase the challenge, you wonder how many ways you could mark off the rest of your answer sheet without ever marking the same letter twice in a row. (For example, if ABD is marked for one question, AC cannot be marked for the next one because A would be marked twice in a row.) If the number of ways to do this can be expressed in the form $2^m p^n$, where $m, n > 1$ are integers and p is a prime, compute $100m + n + p$.

Proposed by Alexander Dai

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5 Zang is at the point $(3, 3)$ in the coordinate plane. Every second, he can move one unit up or one unit right, but he may never visit points where the x and y coordinates are both composite. In how many ways can he reach the point $(20, 13)$?

Based on a proposal by Ahaan Rungta

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6 For each positive integer n, let $H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$. If

$$\sum_{n=4}^{\infty} \frac{1}{nH_nH_{n-1}} = \frac{M}{N}$$

for relatively prime positive integers M and N, compute $100M + N$.

Based on a proposal by ssilwa

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7 In $\triangle ABC$ with $AB = 10$, $AC = 13$, and $\measuredangle ABC = 30^\circ$, M is the midpoint of $\overline{BC}$ and the circle with diameter $\overline{AM}$ meets $\overline{CB}$ and $\overline{CA}$ again at D and E, respectively. The area of $\triangle DEM$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $100m + n$.

Based on a proposal by Matthew Babbitt

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8 Find the number of positive integers n for which there exists a sequence $x_1, x_2, \cdots, x_n$ of integers with the following property: if indices $1 \leq i \leq j \leq n$ satisfy $i + j \leq n$ and $x_i - x_j$ is divisible by 3, then $x_{i+j} + x_i + x_j + 1$ is divisible by 3.

Based on a proposal by Ivan Koswara

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9 Let $ABCD$ be a square of side length 6. Points E and F are selected on rays AB and AD such that segments EF and BC intersect at a point L, D lies between A and F, and the area of $\triangle AEF$ is 36. Clio constructs triangle PQR with $PQ = BL$, $QR = CL$ and $RP = DF$, and notices that the area of $\triangle PQR$ is $\sqrt{6}$. If the sum of all possible values of DF is $\sqrt{m} + \sqrt{n}$ for positive integers $m \ge n$, compute $100m + n$.

Based on a proposal by Calvin Lee

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10 Let $x \neq y$ be positive reals satisfying $x^3 + 2013y = y^3 + 2013x$, and let $M = \left(\sqrt{3}+1\right)x + 2y$. Determine the maximum possible value of M^2.

Proposed by Varun Mohan

Day 7 May 27th

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1 At ARML, Santa is asked to give rubber duckies to 2013 students, one for each student. The students are conveniently numbered $1, 2, \cdots, 2013$, and for any integers $1 \le m < n \le 2013$, students m and n are friends if and only if $0 \le n - 2m \le 1$.

Santa has only four different colors of duckies, but because he wants each student to feel special, he decides to give duckies of different colors to any two students who are either friends or who share a common friend. Let N denote the number of ways in which he can select a color for each student. Find the remainder when N is divided by 1000.

Proposed by Lewis Chen

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2 In $\triangle ABC$, points E and F lie on $\overline{AC}$, $\overline{AB}$, respectively. Denote by P the intersection of $\overline{BE}$ and $\overline{CF}$. Compute the maximum possible area of $\triangle ABC$ if $PB = 14$, $PC = 4$, $PE = 7$, $PF = 2$.

Proposed by Eugene Chen

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3 Richard has a four infinitely large piles of coins: a pile of pennies (worth 1 cent each), a pile of nickels (5 cents), a pile of dimes (10 cents), and a pile of quarters (25 cents). He chooses one pile at random and takes one coin from that pile. Richard then repeats this process until the sum of the values of the coins he has taken is an integer number of dollars. (One dollar is 100 cents.) What is the expected value of this final sum of money, in cents?

Proposed by Lewis Chen

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4 Find the positive integer N for which there exist reals $\alpha, \beta, \gamma, \theta$ which obey

$$\begin{aligned} 0.1 &= \sin\gamma\cos\theta\sin\alpha, \\ 0.2 &= \sin\gamma\sin\theta\cos\alpha, \\ 0.3 &= \cos\gamma\cos\theta\sin\beta, \\ 0.4 &= \cos\gamma\sin\theta\cos\beta, \\ 0.5 &\geq \left|N - 100\cos 2\theta\right|. \end{aligned}$$

Proposed by Evan Chen

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5 For every integer $n \geq 1$, the function $f_n : \{0, 1, \cdots, n\} \to \mathbb{R}$ is defined recursively by $f_n(0) = 0$, $f_n(1) = 1$ and

$$(n-k)f_n(k-1) + kf_n(k+1) = nf_n(k)$$

for each $1 \leq k < n$. Let $S_N = f_{N+1}(1) + f_{N+2}(2) + \cdots + f_{2N}(N)$. Find the remainder when $\lfloor S_{2013} \rfloor$ is divided by 2011. (Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

Proposed by Lewis Chen

Day 8 September 24th

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1 Let a, b, c, d, e be positive reals satisfying

$$\begin{aligned} a + b &= c \\ a + b + c &= d \\ a + b + c + d &= e. \end{aligned}$$

If $c = 5$, compute $a + b + c + d + e$.

Proposed by Evan Chen

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2 A positive integer N has 20 digits when written in base 9 and 13 digits when written in base 27. How many digits does N have when written in base 3?

Proposed by Aaron Lin

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3 Integers a, b, c are selected independently and at random from the set $\{1, 2, \cdots, 10\}$, with replacement. If p is the probability that $a^{b-1}b^{c-1}c^{a-1}$ is a power of two, compute $1000p$.

Proposed by Evan Chen

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4 On side $\overline{AB}$ of square $ABCD$, point E is selected. Points F and G are located on sides $\overline{AB}$ and $\overline{AD}$, respectively, such that $\overline{FG} \perp \overline{CE}$. Let P be the intersection point of segments $\overline{FG}$ and $\overline{CE}$. Given that $[EPF] = 1$, $[EPGA] = 8$, and $[CPFB] = 15$, compute $[PGDC]$. (Here $[\mathcal{P}]$ denotes the area of the polygon $\mathcal{P}$.)

Proposed by Aaron Lin

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5 Let x, y, z be complex numbers satisfying

$$z^2 + 5x = 10z$$
$$y^2 + 5z = 10y$$
$$x^2 + 5y = 10x$$

Find the sum of all possible values of z.

Proposed by Aaron Lin

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6 Let $f(n) = \varphi(n^3)^{-1}$, where $\varphi(n)$ denotes the number of positive integers not greater than n that are relatively prime to n. Suppose

$$\frac{f(1) + f(3) + f(5) + \dots}{f(2) + f(4) + f(6) + \dots} = \frac{m}{n}$$

where m and n are relatively prime positive integers. Compute $100m + n$.

Proposed by Lewis Chen

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7 Dragon selects three positive real numbers with sum 100, uniformly at random. He asks Cat to copy them down, but Cat gets lazy and rounds them all to the nearest tenth during transcription. If the probability the three new numbers still sum to 100 is $\frac{m}{n}$, where m and n are relatively prime positive integers, compute $100m + n$.

Proposed by Aaron Lin

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8 The diagonals of convex quadrilateral $BSCT$ meet at the midpoint M of $\overline{ST}$. Lines BT and SC meet at A, and $AB = 91$, $BC = 98$, $CA = 105$. Given that $\overline{AM} \perp \overline{BC}$, find the positive difference between the areas of $\triangle SMC$ and $\triangle BMT$.

Proposed by Evan Chen

Day 9 November 13th

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1 A sequence $a_0, a_1, a_2, \dots$ of real numbers satisfies $a_0 = 999$, $a_1 = -999$, and $a_n = a_{n-1}a_{n+1}$ for each positive integer n. Compute $|a_1 + a_2 + \dots + a_{1000}|$.

Proposed by Jeremy Lu

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2 Let f be a non-constant polynomial such that

$$f(x-1) + f(x) + f(x+1) = \frac{f(x)^2}{2013x}$$

for all nonzero real numbers x. Find the sum of all possible values of $f(1)$.

Proposed by Ahaan S. Rungta

3 Let $a_1, a_2, \dots, a_{1000}$ be positive integers whose sum is S. If $a_n!$ divides n for each $n = 1, 2, \dots, 1000$, compute the maximum possible value of S.

Proposed by Michael Ren

4 Consider a set of 1001 points in the plane, no three collinear. Compute the minimum number of segments that must be drawn so that among any four points, we can find a triangle.

Proposed by Ahaan S. Rungta / Amir Hossein

5 Let d and n be positive integers such that d divides n, $n > 1000$, and n is not a perfect square. The minimum possible value of $|d - \sqrt{n}|$ can be written in the form $a\sqrt{b} + c$, where b is a positive integer not divisible by the square of any prime, and a and c are nonzero integers (not necessarily positive). Compute $a + b + c$.

Proposed by Matthew Lerner-Brecher

6 Let ABC be a triangle with $AB = 42$, $AC = 39$, $BC = 45$. Let E, F be on the sides $\overline{AC}$ and $\overline{AB}$ such that $AF = 21, AE = 13$. Let $\overline{CF}$ and $\overline{BE}$ intersect at P, and let ray AP meet $\overline{BC}$ at D. Let O denote the circumcenter of $\triangle DEF$, and R its circumradius. Compute $CO^2 - R^2$.

Proposed by Yang Liu

7 Tyler has two calculators, both of which initially display zero. The first calculators has only two buttons, $[+1]$ and $[\times 2]$. The second has only the buttons $[+1]$ and $[\times 4]$. Both calculators update their displays immediately after each keystroke.

A positive integer n is called *ambivalent* if the minimum number of keystrokes needed to display n on the first calculator equals the minimum number of keystrokes needed to display n on the second calculator. Find the sum of all ambivalent integers between 256 and 1024 inclusive.

Proposed by Joshua Xiong

8 Let $ABCD$ be a convex quadrilateral with $\angle ABC = 120^\circ$ and $\angle BCD = 90^\circ$, and let M and N denote the midpoints of $\overline{BC}$ and $\overline{CD}$. Suppose there exists a point P on the circumcircle of $\triangle CMN$ such that ray MP bisects $\overline{AD}$ and ray NP bisects $\overline{AB}$. If $AB + BC = 444$, $CD = 256$ and $BC = \frac{m}{n}$ for some relatively prime positive integers m and n, compute $100m + n$.

Proposed by Michael Ren

Day 10 December 3rd

1 Richard likes to solve problems from the IMO Shortlist. In 2013, Richard solves 5 problems each Saturday and 7 problems each Sunday. He has school on weekdays, so he "only" solves 2, 1, 2, 1, 2 problems on each Monday, Tuesday, Wednesday, Thursday, and Friday, respectively – with the exception of December 3, 2013, where he solved 60 problems out of boredom. Altogether, how many problems does Richard solve in 2013?

Proposed by Evan Chen

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2 How many integers n are there such that $(n+1!)(n+2!)(n+3!)\cdots(n+2013!)$ is divisible by 210 and $1 \le n \le 210$?

Proposed by Lewis Chen

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3 At Stanford in 1988, human calculator Shakuntala Devi was asked to compute $m = \sqrt[3]{61{,}629{,}875}$ and $n = \sqrt[7]{170{,}859{,}375}$. Given that m and n are both integers, compute $100m + n$.

Proposed by Evan Chen

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4 Let $S = \{1, 2, \cdots, 2013\}$. Let N denote the number 9-tuples of sets $(S_1, S_2, \dots, S_9)$ such that $S_{2n-1}, S_{2n+1} \subseteq S_{2n} \subseteq S$ for $n = 1, 2, 3, 4$. Find the remainder when N is divided by 1000.

Proposed by Lewis Chen

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5 In a certain game, Auntie Hall has four boxes B_1, B_2, B_3, B_4, exactly one of which contains a valuable gemstone; the other three contain cups of yogurt. You are told the probability the gemstone lies in box B_n is $\frac{n}{10}$ for $n = 1, 2, 3, 4$.

Initially you may select any of the four boxes; Auntie Hall then opens one of the other three boxes at random (which may contain the gemstone) and reveals its contents. Afterwards, you may change your selection to any of the four boxes, and you win if and only if your final selection contains the gemstone. Let the probability of winning assuming optimal play be $\frac{m}{n}$, where m and n are relatively prime integers. Compute $100m + n$.

Proposed by Evan Chen

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6 Given a regular dodecagon (a convex polygon with 12 congruent sides and angles) with area 1, there are two possible ways to dissect this polygon into 12 equilateral triangles and 6 squares. Let T_1 denote the union of all triangles in the first dissection, and S_1 the union of all squares. Define T_2 and S_2 similarly for the second dissection. Let S and T denote the areas of $S_1 \cap S_2$ and $T_1 \cap T_2$, respectively. If $\frac{S}{T} = \frac{a+b\sqrt{3}}{c}$ where a and b are integers, c is a positive integer, and $\gcd(a, c) = 1$, compute $10000a + 100b + c$.

Proposed by Lewis Chen

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7 Let $ABCD$ be a convex quadrilateral for which $DA = AB$ and $CA = CB$. Set $I_0 = C$ and $J_0 = D$, and for each nonnegative integer n, let I_{n+1} and J_{n+1} denote the incenters of $\triangle I_n AB$

and $\triangle J_nAB$, respectively.
Suppose that

$$\angle DAC = 15^\circ, \quad \angle BAC = 65^\circ \quad \text{and} \quad \angle J_{2013}J_{2014}I_{2014} = \left(90 + \frac{2k+1}{2^n}\right)^\circ$$

for some nonnegative integers n and k. Compute $n+k$.

Proposed by Evan Chen

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8 The number $\frac{1}{2}$ is written on a blackboard. For a real number c with $0 < c < 1$, a [i]c-splay[/i] is an operation in which every number x on the board is erased and replaced by the two numbers cx and $1-c(1-x)$. A *splay-sequence* $C = (c_1, c_2, c_3, c_4)$ is an application of a c_i-splay for $i = 1, 2, 3, 4$ in that order, and its *power* is defined by $P(C) = c_1c_2c_3c_4$.

Let S be the set of splay-sequences which yield the numbers $\frac{1}{17}, \frac{2}{17}, \dots, \frac{16}{17}$ on the blackboard in some order. If $\sum_{C\in S} P(C) = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m+n$.

Proposed by Lewis Chen

– Winter Contest

– March 1st

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1 Find the remainder when $2+4+\cdots+2014$ is divided by $1+3+\cdots+2013$. Justify your answer.

Proposed by Evan Chen

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2 Square $\mathcal{S}$ has vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$. Points P and Q are independently selected, uniformly at random, from the perimeter of $\mathcal{S}$. Determine, with proof, the probability that the slope of line PQ is positive.

Proposed by Isabella Grabski

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3 Let ABC be a triangle. Prove that there exists a unique point P for which one can find points D, E and F such that the quadrilaterals $APBF$, $BPCD$, $CPAE$, $EPFA$, $FPDB$, and $DPEC$ are all parallelograms.

Proposed by Lewis Chen

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4 Let $\mathcal{F}$ be the set of all 2013×2013 arrays whose entries are 0 and 1. A transformation $K : \mathcal{F} \to \mathcal{F}$ is defined as follows: for each entry a_{ij} in an array $A \in \mathcal{F}$, let S_{ij} denote the sum of all the entries of A sharing either a row or column (or both) with a_{ij}. Then a_{ij} is replaced by the remainder when S_{ij} is divided by two.

Prove that for any $A \in \mathcal{F}$, $K(A) = K(K(A))$.

Proposed by Aaron Lin

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5 In convex hexagon $AXBYCZ$, sides AX, BY and CZ are parallel to diagonals BC, XC and XY, respectively. Prove that $\triangle ABC$ and $\triangle XYZ$ have the same area.

Proposed by Evan Chen

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6 A strictly increasing sequence $\{x_i\}_{i=1}^\infty$ of positive integers is said to be *large* if, for every real number L, there exists an integer n such that $\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}>L$. Do there exist large sequences $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$ such that the sequence $\{a_i+b_i\}_{i=1}^\infty$ is not large?

Proposed by Lewis Chen

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7 Let a, b, c be positive reals satisfying $a^3+b^3+c^3+abc=4$. Prove that

$$\frac{(5a^2+bc)^2}{(a+b)(a+c)}+\frac{(5b^2+ca)^2}{(b+c)(b+a)}+\frac{(5c^2+ab)^2}{(c+a)(c+b)}\geq\frac{(a^3+b^3+c^3+6)^2}{a+b+c}$$

and determine the cases of equality.

Proposed by Evan Chen

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8 For a finite set X define

$$S(X)=\sum_{x\in X}x \text{ and } P(x)=\prod_{x\in X}x.$$

Let A and B be two finite sets of positive integers such that $|A|=|B|$, $P(A)=P(B)$ and $S(A)\neq S(B)$. Suppose for any $n\in A\cup B$ and prime p dividing n, we have $p^{36}\mid n$ and $p^{37}\nmid n$. Prove that

$$|S(A)-S(B)|>1.9\cdot 10^6.$$

Proposed by Evan Chen

– April Fun Round

– April 1st

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1 Find the value of 645.

Proposed by George Xing, et al.

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2 At a certain school, the ratio of boys to girls is $1:3$. Suppose that: • Every boy has most 2013 distinct girlfriends. • Every girl has at least n boyfriends. • Friendship is mutual.

Compute the largest possible value of n.

Proposed by Evan Chen

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3 Bored in an infinitely long class, Evan jots down a fraction whose numerator and denominator are both 70-character strings, as follows:

$$r = \frac{looolooll ooll oololll lloloolloll ollll oolll oloolooolo looolololo oooollllol}{lolooloolo llollolloo oooloooloo lolloolllo oollololoo oolllloool olllolooo l}.$$

If $o = 2013$ and $l = \frac{1}{50}$, find $\lceil roll \rceil$.

Proposed by Evan Chen

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4 Let a, b, c be the answers to problems 4, 5, and 6, respectively. In $\triangle ABC$, the measures of $\angle A$, $\angle B$, and $\angle C$ are a, b, c in degrees, respectively. Let D and E be points on segments AB and AC with $\frac{AD}{BD} = \frac{AE}{CE} = 2013$. A point P is selected in the interior of $\triangle ADE$, with barycentric coordinates (x, y, z) with respect to $\triangle ABC$ (here $x + y + z = 1$). Lines BP and CP meet line DE at B_1 and C_1, respectively. Suppose that the radical axis of the circumcircles of $\triangle PDC_1$ and $\triangle PEB_1$ pass through point A. Find $100x$.

Proposed by Evan Chen

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5 Consider $\triangle\natural\flat\sharp$. Let $\flat\sharp$, $\sharp\natural$ and $\natural\flat$ be the answers to problems 4, 5, and 6, respectively. If the incircle of $\triangle\natural\flat\sharp$ touches $\natural\flat$ at $\odot$, find $\flat\odot$.

Proposed by Evan Chen

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6 Let n and k be integers satisfying $\binom{2k}{2} + n = 60$. It is known that n days before Evan's 16th birthday, something happened. Compute $60 - n$.

Proposed by Evan Chen

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7 Let p be the largest prime less than 2013 for which

$$N = 20 + p^{p^{p+1}-13}$$

is also prime. Find the remainder when N is divided by 10^4.

Proposed by Evan Chen and Lewis Chen

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8 A person flips 2010 coins at a time. He gains one penny every time he flips a prime number of heads, but must stop once he flips a non-prime number. If his expected amount of money gained in dollars is $\frac{a}{b}$, where a and b are relatively prime, compute $\lceil \log_2(100a + b) \rceil$.

Proposed by Lewis Chen

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9 Haddaway once asked,"what is love?". The answer can be written in the form $\frac{m}{n}$, where m and n are positive integers such that $m^2 + n^2 < 2013$. Find $100m + n$.

Proposed by Evan Chen

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10 There exist primes p and q such that

$$pq = 1208925819614629174706176 \times 2^{4404} - 4503599560261633 \times 134217730 \times 2^{2202} + 1.$$

Find the remainder when $p+q$ is divided by 1000.

Proposed by Evan Chen

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11 USAYNO: `http://goo.gl/wVR25`

Proposed by Lewis Chen, Evan Chen, Eugene Chen

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12 If X_i is the answer to problem i for $1 \le i \le 12$, find the minimum possible value of $\sum_{n=1}^{12}(-1)^n X_n$.

Proposed by Evan Chen, Lewis Chen

– Summer Contest

– August 10th

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1 What is the maximum possible score on this contest? Recall that on the NIMO 2013 Summer Contest, problems $1, 2, \dots, 15$ are worth $1, 2, \dots, 15$ points, respectively.

Proposed by Evan Chen

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2 If $\frac{2+4+6}{1+3+5} - \frac{1+3+5}{2+4+6} = \frac{m}{n}$ for relatively prime integers m and n, compute $100m+n$.

Proposed by Evan Chen

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3 Jacob and Aaron are playing a game in which Aaron is trying to guess the outcome of an unfair coin which shows heads $\frac{2}{3}$ of the time. Aaron randomly guesses "heads" $\frac{2}{3}$ of the time, and guesses "tails" the other $\frac{1}{3}$ of the time. If the probability that Aaron guesses correctly is p, compute $9000p$.

Proposed by Aaron Lin

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4 Find the sum of the real roots of the polynomial

$$\prod_{k=1}^{100}\left(x^2 - 11x + k\right) = \left(x^2 - 11x + 1\right)\left(x^2 - 11x + 2\right)\dots\left(x^2 - 11x + 100\right).$$

Proposed by Evan Chen

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5 A point (a, b) in the plane is called *sparkling* if it also lies on the line $ax + by = 1$. Find the maximum possible distance between two sparkling points.

Proposed by Evan Chen

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6 Let ABC and DEF be two triangles, such that $AB = DE = 20$, $BC = EF = 13$, and $\angle A = \angle D$. If $AC - DF = 10$, determine the area of $\triangle ABC$.

Proposed by Lewis Chen

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7 Circle ω_1 and ω_2 have centers $(0,6)$ and $(20,0)$, respectively. Both circles have radius 30, and intersect at two points X and Y. The line through X and Y can be written in the form $y = mx+b$. Compute $100m + b$.

Proposed by Evan Chen

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8 A pair of positive integers (m,n) is called *compatible* if $m \ge \frac{1}{2}n + 7$ and $n \ge \frac{1}{2}m + 7$. A positive integer $k \ge 1$ is called *lonely* if (k, ℓ) is not compatible for any integer $\ell \ge 1$. Find the sum of all lonely integers.

Proposed by Evan Chen

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9 Compute $99(99^2 + 3) + 3 \cdot 99^2$.

Proposed by Evan Chen

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10 Let $P(x)$ be the unique polynomial of degree four for which $P(165) = 20$, and

$$P(42) = P(69) = P(96) = P(123) = 13.$$

Compute $P(1) - P(2) + P(3) - P(4) + \cdots + P(165)$.

Proposed by Evan Chen

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11 Find $100m + n$ if m and n are relatively prime positive integers such that

$$\sum_{\substack{i,j \ge 0 \\ i+j \text{ odd}}} \frac{1}{2^i 3^j} = \frac{m}{n}.$$

Proposed by Aaron Lin

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12 In $\triangle ABC$, $AB = 40$, $BC = 60$, and $CA = 50$. The angle bisector of $\angle A$ intersects the circumcircle of $\triangle ABC$ at A and P. Find BP.

Proposed by Eugene Chen

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13 In trapezoid $ABCD$, $AD \parallel BC$ and $\angle ABC + \angle CDA = 270^\circ$. Compute AB^2 given that $AB \cdot \tan(\angle BCD) = 20$ and $CD = 13$.

Proposed by Lewis Chen

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14 Let p, q, and r be primes satisfying

$$pqr = 18\underbrace{9999999999999999999999999999999999999999999999999999}62.$$

Compute $S(p)+S(q)+S(r)-S(pqr)$, where $S(n)$ denote the sum of the decimals digits of n.

Proposed by Evan Chen

15

Ted quite likes haikus,

poems with five-seven-five,

but Ted knows few words.

He knows $2n$ words

that contain n syllables

for every int n.

Ted can only write
N distinct haikus. Find N.

Take mod one hundred.

Ted loves creating haikus (Japanese three-line poems with 5, 7, 5 syllables each), but his vocabulary is rather limited. In particular, for integers $1 \le n \le 7$, he knows $2n$ words with n syllables. Furthermore, words cannot cross between lines, but may be repeated. If Ted can make N distinct haikus, compute the remainder when N is divided by 100.

Proposed by Lewis Chen
