NIMO Problems 2014

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–	Winter Contest
1	Find, with proof, all real numbers x satisfying $x = 2\left(2\left(2\left(2\left(2x-1\right)-1\right)-1\right)-1\right)-1$. *Proposed by Evan Chen*
2	Determine, with proof, the smallest positive integer c such that for any positive integer n, the decimal representation of the number c^n+2014 has digits all less than 5. *Proposed by Evan Chen*
3	The numbers $1, 2, \dots, 10$ are written on a board. Every minute, one can select three numbers a, b, c on the board, erase them, and write $\sqrt{a^2+b^2+c^2}$ in their place. This process continues until no more numbers can be erased. What is the largest possible number that can remain on the board at this point? *Proposed by Evan Chen*
4	Prove that there exist integers a, b, c with $1 \le a < b < c \le 25$ and $$S(a^6+2014) = S(b^6+2014) = S(c^6+2014)$$ where $S(n)$ denotes the sum of the decimal digits of n. *Proposed by Evan Chen*
5	Let ABC be an acute triangle with orthocenter H and let M be the midpoint of $\overline{BC}$. (The *orthocenter* is the point at the intersection of the three altitudes.) Denote by ω_B the circle passing through B, H, and M, and denote by ω_C the circle passing through C, H, and M. Lines AB and AC meet ω_B and ω_C again at P and Q, respectively. Rays PH and QH meet ω_C and ω_B again at R and S, respectively. Show that $\triangle BRS$ and $\triangle CRS$ have the same area. *Proposed by Aaron Lin*
6	Let $\varphi(k)$ denote the numbers of positive integers less than or equal to k and relatively prime to k. Prove that for some positive integer n, $$\varphi(2n-1)+\varphi(2n+1) < \frac{1}{1000}\varphi(2n).$$ *Proposed by Evan Chen*

7 Let ABC be a triangle and let Q be a point such that $\overline{AB} \perp \overline{QB}$ and $\overline{AC} \perp \overline{QC}$. A circle with center I is inscribed in $\triangle ABC$, and is tangent to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ at points D, E, and F, respectively. If ray QI intersects $\overline{EF}$ at P, prove that $\overline{DP} \perp \overline{EF}$.

Proposed by Aaron Lin

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8 Define the function $\xi : \mathbb{Z}^2 \to \mathbb{Z}$ by $\xi(n,k) = 1$ when $n \le k$ and $\xi(n,k) = -1$ when $n > k$, and construct the polynomial

$$P(x_1, \dots, x_{1000}) = \prod_{n=1}^{1000} \left(\sum_{k=1}^{1000} \xi(n,k) x_k \right).$$

(a) Determine the coefficient of $x_1 x_2 \dots x_{1000}$ in P.
(b) Show that if $x_1, x_2, \dots, x_{1000} \in \{-1, 1\}$ then $P(x_1, x_2, \dots, x_{1000}) = 0$.

Proposed by Evan Chen

– Monthly Contests

Day 11 January 31st

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1 Define $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Let the sum of all H_n that are terminating in base 10 be S. If $S = m/n$ where m and n are relatively prime positive integers, find $100m + n$.

Proposed by Lewis Chen

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2 In the game of Guess the Card, two players each have a $\frac{1}{2}$ chance of winning and there is exactly one winner. Sixteen competitors stand in a circle, numbered $1, 2, \dots, 16$ clockwise. They participate in an 4-round single-elimination tournament of Guess the Card. Each round, the referee randomly chooses one of the remaining players, and the players pair off going clockwise, starting from the chosen one; each pair then plays Guess the Card and the losers leave the circle. If the probability that players 1 and 9 face each other in the last round is $\frac{m}{n}$ where m, n are positive integers, find $100m + n$.

Proposed by Evan Chen

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3 Call an integer k *debatable* if the number of odd factors of k is a power of two. What is the largest positive integer n such that there exists n consecutive debatable numbers? (Here, a power of two is defined to be any number of the form 2^m, where m is a nonnegative integer.)

Proposed by Lewis Chen

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4 Let a, b, c be positive reals for which

$$(a+b)(a+c) = bc + 2$$
$$(b+c)(b+a) = ca + 5$$
$$(c+a)(c+b) = ab + 9$$

If $abc = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by Evan Chen

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5 In triangle ABC, $\sin A \sin B \sin C = \frac{1}{1000}$ and $AB \cdot BC \cdot CA = 1000$. What is the area of triangle ABC?

Proposed by Evan Chen

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6 Suppose we wish to pick a random integer between 1 and N inclusive by flipping a fair coin. One way we can do this is through generating a random binary decimal between 0 and 1, then multiplying the result by N and taking the ceiling. However, this would take an infinite amount of time. We therefore stopping the flipping process after we have enough flips to determine the ceiling of the number. For instance, if $N = 3$, we could conclude that the number is 2 after flipping $.011_2$, but $.010_2$ is inconclusive.

Suppose $N = 2014$. The expected number of flips for such a process is $\frac{m}{n}$ where m, n are relatively prime positive integers, find $100m + n$.

Proposed by Lewis Chen

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7 Let $P(n)$ be a polynomial of degree m with integer coefficients, where $m \le 10$. Suppose that $P(0) = 0$, $P(n)$ has m distinct integer roots, and $P(n) + 1$ can be factored as the product of two nonconstant polynomials with integer coefficients. Find the sum of all possible values of $P(2)$.

Proposed by Evan Chen

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8 The side lengths of $\triangle ABC$ are integers with no common factor greater than 1. Given that $\angle B = 2\angle C$ and $AB < 600$, compute the sum of all possible values of AB.

Proposed by Eugene Chen

Day 12 February 24th

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1 You drop a 7 cm long piece of mechanical pencil lead on the oor. A bully takes the lead and breaks it at a random point into two pieces. A piece of lead is unusable if it is 2 cm or shorter. If the expected value of the number of usable pieces afterwards is $\frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by Aaron Lin

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2 Let ABC be an equilateral triangle. Denote by D the midpoint of $\overline{BC}$, and denote the circle with diameter $\overline{AD}$ by Ω. If the region inside Ω and outside $\triangle ABC$ has area $800\pi - 600\sqrt{3}$, find the length of AB.

Proposed by Eugene Chen

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3 In land of Nyemo, the unit of currency is called a *quack*. The citizens use coins that are worth 1, 5, 25, and 125 quacks. How many ways can someone pay o 125 quacks using these coins?

Proposed by Aaron Lin

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4 Let S be the set of integers which are both a multiple of 70 and a factor of $630{,}000$. A random element c of S is selected. If the probability that there exists an integer d with $\gcd(c, d) = 70$ and $\operatorname{lcm}(c, d) = 630{,}000$ is $\frac{m}{n}$ for some relatively prime integers m and n, compute $100m + n$.

Proposed by Eugene Chen

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5 Triangle ABC has sidelengths $AB = 14, BC = 15,$ and $CA = 13$. We draw a circle with diameter AB such that it passes BC again at D and passes CA again at E. If the circumradius of $\triangle CDE$ can be expressed as $\frac{m}{n}$ where m, n are coprime positive integers, determine $100m + n$.

Proposed by Lewis Chen

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6 Let $N = 10^6$. For which integer a with $0 \le a \le N - 1$ is the value of

$$\binom{N}{a+1} - \binom{N}{a}$$

maximized?

Proposed by Lewis Chen

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7 Find the sum of all integers n with $2 \le n \le 999$ and the following property: if x and y are randomly selected without replacement from the set $\{1, 2, \dots, n\}$, then $x + y$ is even with probability p, where p is the square of a rational number.

Proposed by Ivan Koswara

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8 Let a, b, c, d be complex numbers satisfying

$$\begin{aligned} 5 &= a + b + c + d \\ 125 &= (5-a)^4 + (5-b)^4 + (5-c)^4 + (5-d)^4 \\ 1205 &= (a+b)^4 + (b+c)^4 + (c+d)^4 + (d+a)^4 + (a+c)^4 + (b+d)^4 \\ 25 &= a^4 + b^4 + c^4 + d^4 \end{aligned}$$

Compute $abcd$.

Proposed by Evan Chen

Day 13 March 24th

1 Let $\eta(m)$ be the product of all positive integers that divide m, including 1 and m. If $\eta(\eta(\eta(10))) = 10^n$, compute n.

Proposed by Kevin Sun

2 Two points A and B are selected independently and uniformly at random along the perimeter of a unit square with vertices at $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. The probability that the y-coordinate of A is strictly greater than the y-coordinate of B can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $100m+n$.

Proposed by Rajiv Movva

3 Find the number of positive integers n with exactly 1974 factors such that no prime greater than 40 divides n, and n ends in one of the digits 1, 3, 7, 9. (Note that $1974 = 2\cdot 3\cdot 7\cdot 47$.)

Proposed by Yonah Borns-Weil

4 A black bishop and a white king are placed randomly on a 2000×2000 chessboard (in distinct squares). Let p be the probability that the bishop attacks the king (that is, the bishop and king lie on some common diagonal of the board). Then p can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute m.

Proposed by Ahaan Rungta

5 Let a positive integer n be *nice* if there exists a positive integer m such that

$$n^3 < 5mn < n^3 + 100.$$

Find the number of *nice* positive integers.

Proposed by Akshaj

6 Let $P(x)$ be a polynomial with real coefficients such that $P(12) = 20$ and

$$(x-1)\cdot P(16x) = (8x-1)\cdot P(8x)$$

holds for all real numbers x. Compute the remainder when $P(2014)$ is divided by 1000.

Proposed by Alex Gu

7 Let N denote the number of ordered pairs of sets (A, B) such that $A \cup B$ is a size-999 subset of $\{1, 2, \dots, 1997\}$ and $(A \cap B) \cap \{1, 2\} = \{1\}$. If m and k are integers such that $3^m 5^k$ divides N, compute the the largest possible value of $m + k$.

Proposed by Michael Tang

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8 Triangle ABC lies entirely in the first quadrant of the Cartesian plane, and its sides have slopes 63, 73, 97. Suppose the curve $\mathcal{V}$ with equation $y = (x + 3)(x^2 + 3)$ passes through the vertices of ABC. Find the sum of the slopes of the three tangents to $\mathcal{V}$ at each of A, B, C.

Proposed by Akshaj

Day 14 May 15th

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1 Let A, B, C, D be four points on a line in this order. Suppose that $AC = 25$, $BD = 40$, and $AD = 57$. Compute $AB \cdot CD + AD \cdot BC$.

Proposed by Evan Chen

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2 In the Generic Math Tournament, 99 people participate. One of the participants, Alfred, scores 16th in Algebra, 30th in Combinatorics, and 23rd in Geometry (and does not tie with anyone). The overall ranking is computed by adding the scores from all three tests. Given this information, let B be the best ranking that Alfred could have achieved, and let W be the worst ranking that he could have achieved. Compute $100B + W$.

Proposed by Lewis Chen

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3 In triangle ABC, we have $AB = AC = 20$ and $BC = 14$. Consider points M on $\overline{AB}$ and N on $\overline{AC}$. If the minimum value of the sum $BN + MN + MC$ is x, compute $100x$.

Proposed by Lewis Chen

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4 Define the infinite products

$$A = \prod_{i=2}^{\infty} \left(1 - \frac{1}{n^3}\right) \text{ and } B = \prod_{i=1}^{\infty} \left(1 + \frac{1}{n(n+1)}\right).$$

If $\frac{A}{B} = \frac{m}{n}$ where m, n are relatively prime positive integers, determine $100m + n$.

Proposed by Lewis Chen

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5 Find the largest integer n for which 2^n divides

$$\binom{2}{1}\binom{4}{2}\binom{6}{3}\dots\binom{128}{64}.$$

Proposed by Evan Chen

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6 10 students are arranged in a row. Every minute, a new student is inserted in the row (which can occur in the front and in the back as well, hence 11 possible places) with a uniform $\frac{1}{11}$ probability of each location. Then, either the frontmost or the backmost student is removed from the row (each with a $\frac{1}{2}$ probability).

Suppose you are the eighth in the line from the front. The probability that you exit the row from the front rather than the back is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Lewis Chen

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7 Ana and Banana play a game. First, Ana picks a real number p with $0 \le p \le 1$. Then, Banana picks an integer h greater than 1 and creates a spaceship with h hit points. Now every minute, Ana decreases the spaceship's hit points by 2 with probability $1 - p$, and by 3 with probability p. Ana wins if and only if the number of hit points is reduced to exactly 0 at some point (in particular, if the spaceship has a negative number of hit points at any time then Ana loses). Given that Ana and Banana select p and h optimally, compute the integer closest to $1000p$.

Proposed by Lewis Chen

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8 Let x be a positive real number. Define

$$A = \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!}, \quad B = \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)!}, \quad \text{and} \quad C = \sum_{k=0}^{\infty} \frac{x^{3k+2}}{(3k+2)!}.$$

Given that $A^3 + B^3 + C^3 + 8ABC = 2014$, compute ABC.

Proposed by Evan Chen

Day 15 September 14th

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1 Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let D be the point inside triangle ABC with the property that $\overline{BD} \perp \overline{CD}$ and $\overline{AD} \perp \overline{BC}$. Then the length AD can be expressed in the form $m - \sqrt{n}$, where m and n are positive integers. Find $100m + n$.

Proposed by Michael Ren

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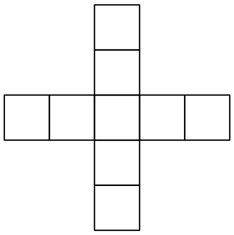
2 In the figure below, how many ways are there to select two squares which do not share an edge?



Proposed by Evan Chen

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3 Let $S = \{1, 2, \ldots, 2014\}$. Suppose that

$$\sum_{T \subseteq S} i^{|T|} = p + qi$$

where p and q are integers, $i = \sqrt{-1}$, and the summation runs over all 2^{2014} subsets of S. Find the remainder when $|p| + |q|$ is divided by 1000. (Here $|X|$ denotes the number of elements in a set X.)

Proposed by David Altizio

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4 Points A, B, C, and D lie on a circle such that chords $\overline{AC}$ and $\overline{BD}$ intersect at a point E inside the circle. Suppose that $\angle ADE = \angle CBE = 75^\circ$, $BE = 4$, and $DE = 8$. The value of AB^2 can be written in the form $a + b\sqrt{c}$ for positive integers a, b, and c such that c is not divisible by the square of any prime. Find $a + b + c$.

Proposed by Tony Kim

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5 Let r, s, t be the roots of the polynomial $x^3 + 2x^2 + x - 7$. Then

$$\left(1 + \frac{1}{(r+2)^2}\right)\left(1 + \frac{1}{(s+2)^2}\right)\left(1 + \frac{1}{(t+2)^2}\right) = \frac{m}{n}$$

for relatively prime positive integers m and n. Compute $100m + n$.

Proposed by Justin Stevens

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6 For all positive integers k, define $f(k) = k^2 + k + 1$. Compute the largest positive integer n such that

$$2015 f(1^2) f(2^2) \cdots f(n^2) \geq \Big(f(1) f(2) \cdots f(n)\Big)^2.$$

Proposed by David Altizio

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7 Find the sum of the prime factors of 67208001, given that 23 is one.

Proposed by Justin Stevens

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8 For positive integers a, b, and c, define

$$f(a,b,c) = \frac{abc}{\text{gcd}(a,b,c) \cdot \text{lcm}(a,b,c)}.$$

We say that a positive integer n is $f@$ if there exist pairwise distinct positive integers $x, y, z \leq 60$ that satisfy $f(x,y,z) = n$. How many $f@$ integers are there?

Proposed by Michael Ren

Day 16

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1 For any interval $\mathcal{A}$ in the real number line not containing zero, define its *reciprocal* to be the set of numbers of the form $\frac{1}{x}$ where x is an element in $\mathcal{A}$. Compute the number of ordered pairs of positive integers (m,n) with $m < n$ such that the length of the interval $[m,n]$ is 10^{10} times the length of its reciprocal.

Proposed by David Altizio

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2 Let $0^\circ \leq \alpha, \beta, \gamma \leq 90^\circ$ be angles such that

$$\sin\alpha - \cos\beta = \tan\gamma$$

$$\sin\beta - \cos\alpha = \cot\gamma$$

Compute the sum of all possible values of γ in degrees.

Proposed by Michael Ren

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3 Let $ABCD$ be a square with side length 2. Let M and N be the midpoints of $\overline{BC}$ and $\overline{CD}$ respectively, and let X and Y be the feet of the perpendiculars from A to $\overline{MD}$ and $\overline{NB}$, also respectively. The square of the length of segment $\overline{XY}$ can be written in the form $\frac{p}{q}$ where p and q are positive relatively prime integers. What is $100p+q$?

Proposed by David Altizio

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4 Let a and b be positive real numbers such that $ab = 2$ and

$$\frac{a}{a+b^2} + \frac{b}{b+a^2} = \frac{7}{8}.$$

Find $a^6 + b^6$.

Proposed by David Altizio

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5 A positive integer N greater than 1 is described as special if in its base-8 and base-9 representations, both the leading and ending digit of N are equal to 1. What is the smallest special integer in decimal representation?

Proposed by Michael Ren

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6 Bob is making partitions of 10, but he hates even numbers, so he splits 10 up in a special way. He starts with 10, and at each step he takes every even number in the partition and replaces it with a random pair of two smaller positive integers that sum to that even integer. For example, 6 could be replaced with $1+5$, $2+4$, or $3+3$ all with equal probability. He terminates this process when all the numbers in his list are odd. The expected number of integers in his list at the end can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $100m+n$.

Proposed by Michael Ren

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7 Let $\triangle ABC$ have $AB=6$, $BC=7$, and $CA=8$, and denote by ω its circumcircle. Let N be a point on ω such that AN is a diameter of ω. Furthermore, let the tangent to ω at A intersect BC at T, and let the second intersection point of NT with ω be X. The length of $\overline{AX}$ can be written in the form $\frac{m}{\sqrt{n}}$ for positive integers m and n, where n is not divisible by the square of any prime. Find $100m+n$.

Proposed by David Altizio

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8 Let $p = 2^{16}+1$ be a prime. A sequence of 2^{16} positive integers $\{a_n\}$ is *monotonically bounded* if $1 \le a_i \le i$ for all $1 \le i \le 2^{16}$. We say that a term a_k in the sequence with $2 \le k \le 2^{16}-1$ is a *mountain* if a_k is greater than both a_{k-1} and a_{k+1}. Evan writes out all possible monotonically bounded sequences. Let N be the total number of mountain terms over all such sequences he writes. Find the remainder when N is divided by p.

Proposed by Michael Ren

– April Fool's

– April 1st

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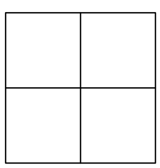
1 How many ways are there to fill the 2×2 grid below with 0's and 1's such that no row or column has duplicate entries?



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2 I'm thinking of a five-letter word that rhymes with “angry” and “hungry”. What is it?

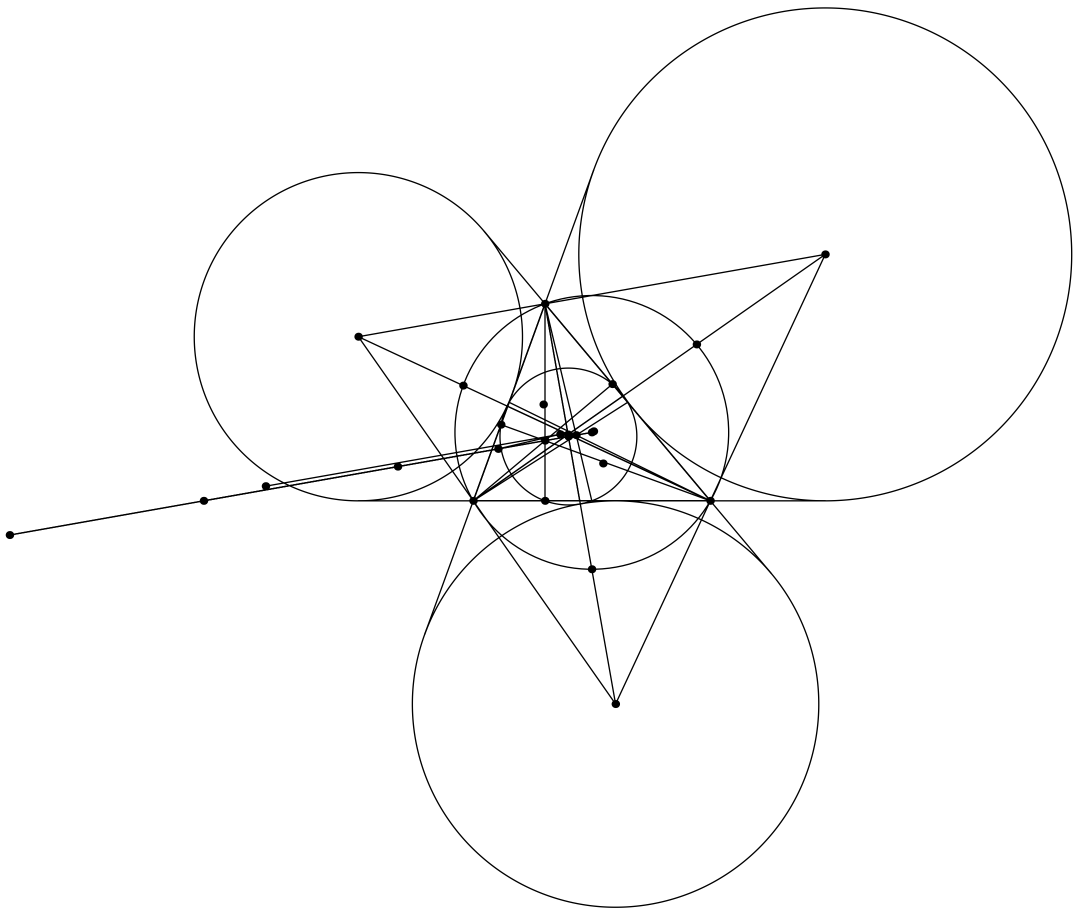
3

4 Let n be largest number such that

$$\frac{2014^{100!} - 2011^{100!}}{3^n}$$

is still an integer. Compute the remainder when 3^n is divided by 1000.

5 Let ABC be a triangle with $AB = 130$, $BC = 140$, $CA = 150$. Let G, H, I, O, N, K, L be the centroid, orthocenter, incenter, circumenter, nine-point center, the symmedian point, and the de Longchamps point. Let D, E, F be the feet of the altitudes of A, B, C on the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$. Let X, Y, Z be the A, B, C excenters and let U, V, W denote the midpoints of $\overline{IX}$, $\overline{IY}$, $\overline{IZ}$ (i.e. the midpoints of the arcs of (ABC).) Let R, S, T denote the isogonal conjugates of the midpoints of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Let P and Q denote the images of G and H under an inversion around the circumcircle of ABC followed by a dilation at O with factor $\frac{1}{2}$, and denote by M the midpoint of $\overline{PQ}$. Then let J be a point such that $JKLM$ is a parallelogram. Find the perimeter of the convex hull of the self-intersecting 17-gon $LETSTRADEBITCOINS$ to the nearest integer. A diagram has been included but may not be to scale.



6 We know $\mathbb{Z}_{210} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7$.
Moreover,

$$53 \equiv 1 \pmod 2$$
$$53 \equiv 2 \pmod 3$$
$$53 \equiv 3 \pmod 5$$
$$53 \equiv 4 \pmod 7.$$

Let

$$M = \begin{pmatrix} 53 & 158 & 53 \\ 23 & 93 & 53 \\ 50 & 170 & 53 \end{pmatrix}.$$

Based on the above, find $\overline{(M \mod 2)(M \mod 3)(M \mod 5)(M \mod 7)}$.

7 Evaluate the following: `http://internetolympiad.org/archive/2014/AprilFools/foreign_lang.txt`

8 Three of the below entries, with labels a, b, c, are blatantly incorrect (in the United States).

What is $a^2 + b^2 + c^2$?

041. The Gentleman's Alliance Cross
042. Glutamine (an amino acid)
051. Grant Nelson and Norris Windross
052. A compact region at the center of a galaxy
061. The value of `'wat'-1`. (See `https://www.destroyallsoftware.com/talks/wat`.)
062. Threonine (an amino acid)
071. Nintendo Gamecube
072. Methane and other gases are compressed
081. A prank or trick
082. Three carbons
091. Australia's second largest local government area
092. Angoon Seaplane Base
101. A compressed archive file format
102. Momordica cochinchinensis
111. Gentaro Takahashi
112. Nat Geo
121. Ante Christum Natum
122. The supreme Siberian god of death
131. Gnu C Compiler
132. My TeX Shortcut for $\angle$.

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9 This is an ARML Super Relay! I'm sure you know how this works! You start from#1 and#15 and meet in the middle.
We are going to require you to solve all 15 problems, though – so for the entire task, submit the sum of all the answers, rather than just the answer to#8.

Also, uhh, we can't actually find the slip for#1. Sorry about that. Have fun anyways!

Problem 2.
Let $T = TNYWR$. Find the number of way to distribute 6 indistinguishable pieces of candy to T hungry (and distinguishable) schoolchildren, such that each child gets at most one piece of candy.

Problem 3.
Let $T = TNYWR$. If d is the largest proper divisor of T, compute $\frac{1}{2}d$.

Problem 4.
Let $T = TNYWR$ and flip 4 fair coins. Suppose the probability that at most T heads appear is $\frac{m}{n}$, where m and n are coprime positive integers. Compute $m + n$.

Problem 5.
Let $T = TNYWR$. Compute the last digit of T^T in base 10.

Problem 6.
Let $T = TNYWR$ and flip 6 fair coins. Suppose the probability that at most T heads appear is $\frac{m}{n}$, where m and n are coprime positive integers. Compute $m+n$.

Problem 7.
Let $T = TNYWR$. Compute the smallest prime p for which $n^T \not\equiv n \pmod{p}$ for some integer n.

Problem 8.
Let M and N be the two answers received, with $M \le N$. Compute the number of integer quadruples (w,x,y,z) with $w+x+y+z = M\sqrt{wxyz}$ and $1 \le w,x,y,z \le N$.

Problem 9.
Let $T = TNYWR$. Compute the smallest integer n with $n \ge 2$ such that n is coprime to $T+1$, and there exists positive integers a, b, c with $a^2+b^2+c^2 = n(ab+bc+ca)$.

Problem 10.
Let $T = TNYWR$ and flip 10 fair coins. Suppose the probability that at most T heads appear is $\frac{m}{n}$, where m and n are coprime positive integers. Compute $m+n$.

Problem 11.
Let $T = TNYWR$. Compute the last digit of T^T in base 10.

Problem 12.
Let $T = TNYWR$ and flip 12 fair coins. Suppose the probability that at most T heads appear is $\frac{m}{n}$, where m and n are coprime positive integers. Compute $m+n$.

Problem 13.
Let $T = TNYWR$. If d is the largest proper divisor of T, compute $\frac{1}{2}d$.

Problem 14.
Let $T = TNYWR$. Compute the number of way to distribute 6 indistinguishable pieces of candy to T hungry (and distinguishable) schoolchildren, such that each child gets at most one piece of candy.

Also, we can't find the slip for#15, either. We think the SFBA coaches stole it to prevent us from winning the Super Relay, but that's not going to stop us, is it? We have another#15 slip that produces an equivalent answer. Here you go!

Problem 15.
Let A, B, C be the answers to#8,#9,#10. Compute $\gcd(A,C)\cdot B$.

– Summer Contest

– August 23rd

1 Compute $1+2\cdot 3^4$.

Proposed by Evan Chen

2 How many $2\times 2\times 2$ cubes must be added to a $8\times 8\times 8$ cube to form a $12\times 12\times 12$ cube?

Proposed by Evan Chen

3 A square and equilateral triangle have the same perimeter. If the triangle has area $16\sqrt{3}$, what is the area of the square?

Proposed by Evan Chen

4 Let n be a positive integer. Determine the smallest possible value of $1-n+n^2-n^3+\dots+n^{1000}$.

Proposed by Evan Chen

5 We have a five-digit positive integer N. We select every pair of digits of N (and keep them in order) to obtain the $\binom{5}{2}=10$ numbers 33, 37, 37, 37, 38, 73, 77, 78, 83, 87. Find N.

Proposed by Lewis Chen

6 Suppose x is a random real number between 1 and 4, and y is a random real number between 1 and 9. If the expected value of

$$\lceil \log_2 x \rceil - \lfloor \log_3 y \rfloor$$

can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers, compute $100m+n$.

Proposed by Lewis Chen

7 Evaluate

$$\frac{1}{729}\sum_{a=1}^{9}\sum_{b=1}^{9}\sum_{c=1}^{9}\left(abc+ab+bc+ca+a+b+c\right).$$

Proposed by Evan Chen

8 Aaron takes a square sheet of paper, with one corner labeled A. Point P is chosen at random inside of the square and Aaron folds the paper so that points A and P coincide. He cuts the sheet along the crease and discards the piece containing A. Let p be the probability that the remaining piece is a pentagon. Find the integer nearest to $100p$.

Proposed by Aaron Lin

9 Two players play a game involving an $n\times n$ grid of chocolate. Each turn, a player may either eat a piece of chocolate (of any size), or split an existing piece of chocolate into two rectangles along a grid-line. The player who moves last loses. For how many positive integers n less than 1000 does the second player win?

(Splitting a piece of chocolate refers to taking an $a \times b$ piece, and breaking it into an $(a-c) \times b$ and a $c \times b$ piece, or an $a \times (b-d)$ and an $a \times d$ piece.)

Proposed by Lewis Chen

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10 Among 100 points in the plane, no three collinear, exactly 4026 pairs are connected by line segments. Each point is then randomly assigned an integer from 1 to 100 inclusive, each equally likely, such that no integer appears more than once. Find the expected value of the number of segments which join two points whose labels differ by at least 50.

Proposed by Evan Chen

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11 Consider real numbers A, B, $\dots$, Z such that

$$EVIL = \frac{5}{31}, \; LOVE = \frac{6}{29}, \text{ and } IMO = \frac{7}{3}.$$

If $OMO = \tfrac{m}{n}$ for relatively prime positive integers m and n, find the value of $m+n$.

Proposed by Evan Chen

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12 Find the sum of all positive integers n such that

$$\frac{2n+1}{n(n-1)}$$

has a terminating decimal representation.

Proposed by Evan Chen

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13 Let α and β be nonnegative integers. Suppose the number of strictly increasing sequences of integers $a_0, a_1, \dots, a_{2014}$ satisfying $0 \le a_m \le 3m$ is $2^\alpha(2\beta+1)$. Find α.

Proposed by Lewis Chen

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14 Let ABC be a triangle with circumcenter O and let X, Y, Z be the midpoints of arcs BAC, ABC, ACB on its circumcircle. Let G and I denote the centroid of $\triangle XYZ$ and the incenter of $\triangle ABC$.

Given that $AB = 13$, $BC = 14$, $CA = 15$, and $\frac{GO}{GI} = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by Evan Chen

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15 Let $A = (0, 0)$, $B = (-1, -1)$, $C = (x, y)$, and $D = (x + 1, y)$, where $x > y$ are positive integers. Suppose points A, B, C, D lie on a circle with radius r. Denote by r_1 and r_2 the smallest and second smallest possible values of r. Compute $r_1^2 + r_2^2$.

Proposed by Lewis Chen