NIMO Problems 2015

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by djmathman, Binomial-theorem

Day 17 January 22nd

1 Let $2000 < N < 2100$ be an integer. Suppose the last day of year N is a Tuesday while the first day of year $N+2$ is a Friday. The fourth Sunday of year $N+3$ is the mth day of January. What is m?

Based on a proposal by Neelabh Deka

2 Let $ABCD$ be a square with side length 100. Denote by M the midpoint of AB. Point P is selected inside the square so that $MP = 50$ and $PC = 100$. Compute AP^2.

Based on a proposal by Amogh Gaitonde

3 How many 5-digit numbers N (in base 10) contain no digits greater than 3 and satisfy the equality $\gcd(N, 15) = \gcd(N, 20) = 1$? (The leading digit of N cannot be zero.)

Based on a proposal by Yannick Yao

4 Determine the number of positive integers $a \le 250$ for which the set $\{a+1, a+2, \dots, a+1000\}$ contains

• Exactly 333 multiples of 3, • Exactly 142 multiples of 7, and • Exactly 91 multiples of 11.

Based on a proposal by Rajiv Movva

5 Let $a, b, c, d, e,$ and f be real numbers. Define the polynomials

$$P(x) = 2x^4 - 26x^3 + ax^2 + bx + c \quad \text{and} \quad Q(x) = 5x^4 - 80x^3 + dx^2 + ex + f.$$

Let S be the set of all complex numbers which are a root of *either* P or Q (or both). Given that $S = \{1, 2, 3, 4, 5\}$, compute $P(6) \cdot Q(6)$.

Proposed by Michael Tang

6 Let $\triangle ABC$ be a triangle with $BC = 4, CA = 5, AB = 6$, and let O be the circumcenter of $\triangle ABC$. Let O_b and O_c be the reflections of O about lines CA and AB respectively. Suppose BO_b and CO_c intersect at T, and let M be the midpoint of BC. Given that $MT^2 = \frac{p}{q}$ for some coprime positive integers p and q, find $p+q$.

Proposed by Sreejato Bhattacharya

7 Find the number of ways a series of $+$ and $-$ signs can be inserted between the numbers $0, 1, 2, \cdots, 12$ such that the value of the resulting expression is divisible by 5.

Proposed by Matthew Lerner-Brecher

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8 For an integer $30 \le k \le 70$, let M be the maximum possible value of

$$\frac{A}{\gcd(A, B)} \quad \text{where } A = \binom{100}{k} \text{ and } B = \binom{100}{k+3}.$$

Find the remainder when M is divided by 1000.

Based on a proposal by Michael Tang

Day 18 March 22nd

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1 A function f from the positive integers to the nonnegative integers is defined recursively by $f(1) = 0$ and $f(n+1) = 2^{f(n)}$ for every positive integer n. What is the smallest n such that $f(n)$ exceeds the number of atoms in the observable universe (approximately 10^{80})?

Proposed by Evan Chen

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2 There exists a unique strictly increasing arithmetic sequence $\{a_i\}_{i=1}^{100}$ of positive integers such that

$$a_1 + a_4 + a_9 + \cdots + a_{100} = 1000,$$

where the summation runs over all terms of the form a_{i^2} for $1 \le i \le 10$. Find a_{50}.

Proposed by David Altizio and Tony Kim

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3 Let $ABCD$ be a rectangle with $AB = 6$ and $BC = 6\sqrt{3}$. We construct four semicircles ω_1, ω_2, ω_3, ω_4 whose diameters are the segments AB, BC, CD, DA. It is given that ω_i and ω_{i+1} intersect at some point X_i in the interior of $ABCD$ for every $i = 1, 2, 3, 4$ (indices taken modulo 4). Compute the square of the area of $X_1X_2X_3X_4$.

Proposed by Evan Chen

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4 Find the sum of all positive integers $1 \le k \le 99$ such that there exist positive integers a and b with the property that

$$x^{100} - ax^k + b = (x^2 - 2x + 1)P(x)$$

for some polynomial P with integer coefficients.

Proposed by David Altizio

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5 Compute the number of subsets S of $\{0, 1, \dots, 14\}$ with the property that for each $n = 0, 1, \dots, 6$, either n is in S or both of $2n+1$ and $2n+2$ are in S.

Proposed by Evan Chen

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6 Let ABC be a triangle with $AB = 5$, $BC = 7$, and $CA = 8$. Let D be a point on BC, and define points B' and C' on line AD (or its extension) such that $BB' \perp AD$ and $CC' \perp AD$. If $B'A = B'C'$, then the ratio $BD : DC$ can be expressed in the form $m : n$, where m and n are relatively prime positive integers. Compute $100m + n$.

Proposed by Michael Ren

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7 In a 4×4 grid of unit squares, five squares are chosen at random. The probability that no two chosen squares share a side is $\frac{m}{n}$ for positive relatively prime integers m and n. Find $m + n$.

Proposed by David Altizio

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8 Let ABC be a non-degenerate triangle with incenter I and circumcircle Γ. Denote by M_a the midpoint of the arc $\widehat{BC}$ of Γ not containing A, and define M_b, M_c similarly. Suppose $\triangle ABC$ has inradius 4 and circumradius 9. Compute the maximum possible value of

$$IM_a^2 + IM_b^2 + IM_c^2.$$

Proposed by David Altizio

Day 19 May 19th

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1 Let Ω_1 and Ω_2 be two circles in the plane. Suppose the common external tangent to Ω_1 and Ω_2 has length 2017 while their common internal tangent has length 2009. Find the product of the radii of Ω_1 and Ω_2.

Proposed by David Altizio

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2 Consider the set S of the eight points (x, y) in the Cartesian plane satisfying $x, y \in \{-1, 0, 1\}$ and $(x, y) \neq (0, 0)$. How many ways are there to draw four segments whose endpoints lie in S such that no two segments intersect, even at endpoints?

Proposed by Evan Chen

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3 Let O, A, B, and C be points in space such that $\angle AOB = 60^\circ$, $\angle BOC = 90^\circ$, and $\angle COA = 120^\circ$. Let θ be the acute angle between planes AOB and AOC. Given that $\cos^2\theta = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by Michael Ren

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4 Let $A_0A_1 \dots A_{11}$ be a regular 12-gon inscribed in a circle with diameter 1. For how many subsets $S \subseteq \{1, \dots, 11\}$ is the product

$$\prod_{s \in S} A_0A_s$$

equal to a rational number? (The empty product is declared to be 1.)

Proposed by Evan Chen

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5 Let a, b, c be positive integers and p be a prime number. Assume that

$$a^n(b+c)+b^n(a+c)+c^n(a+b)\equiv 8 \pmod{p}$$

for each nonnegative integer n. Let m be the remainder when $a^p+b^p+c^p$ is divided by p, and k the remainder when m^p is divided by p^4. Find the maximum possible value of k.

Proposed by Justin Stevens and Evan Chen
